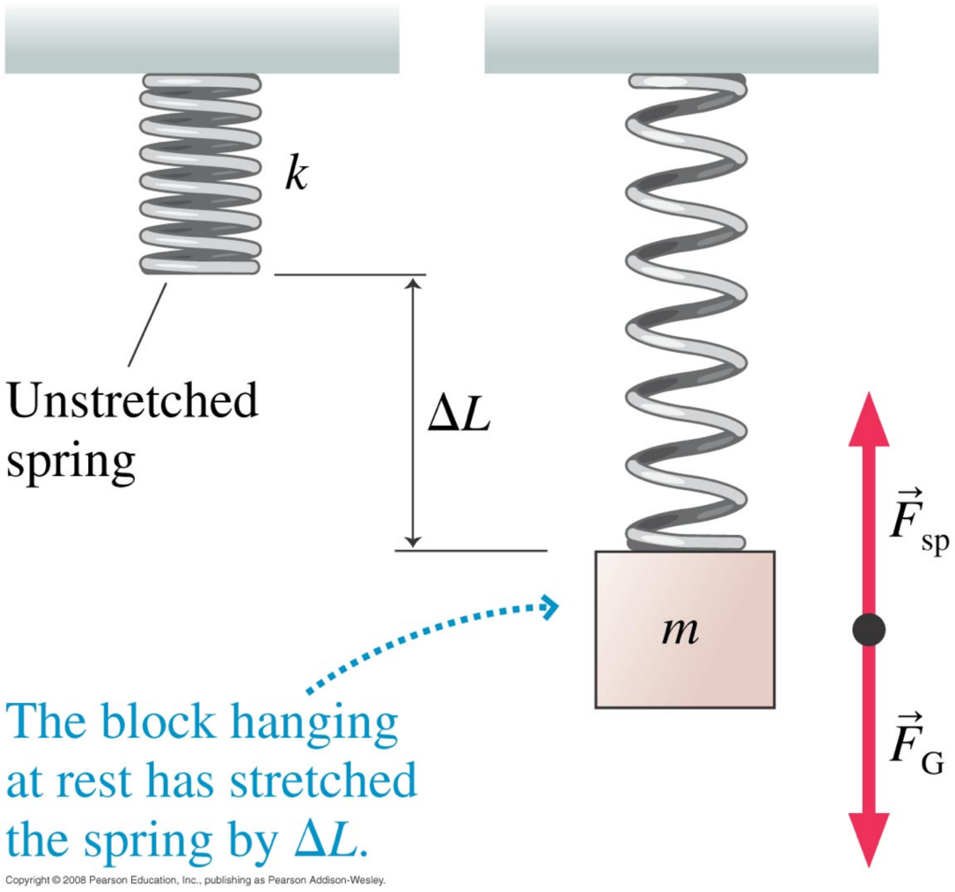
Lecture 2

Outline

- Vertical oscillations of mass on spring
- Pendulum
- Damped and Driven oscillations (more realistic)

Vertical Oscillations (I)

- At equilibrium (no net force), spring is stretched (cf. horizontal spring): spring force balances gravity



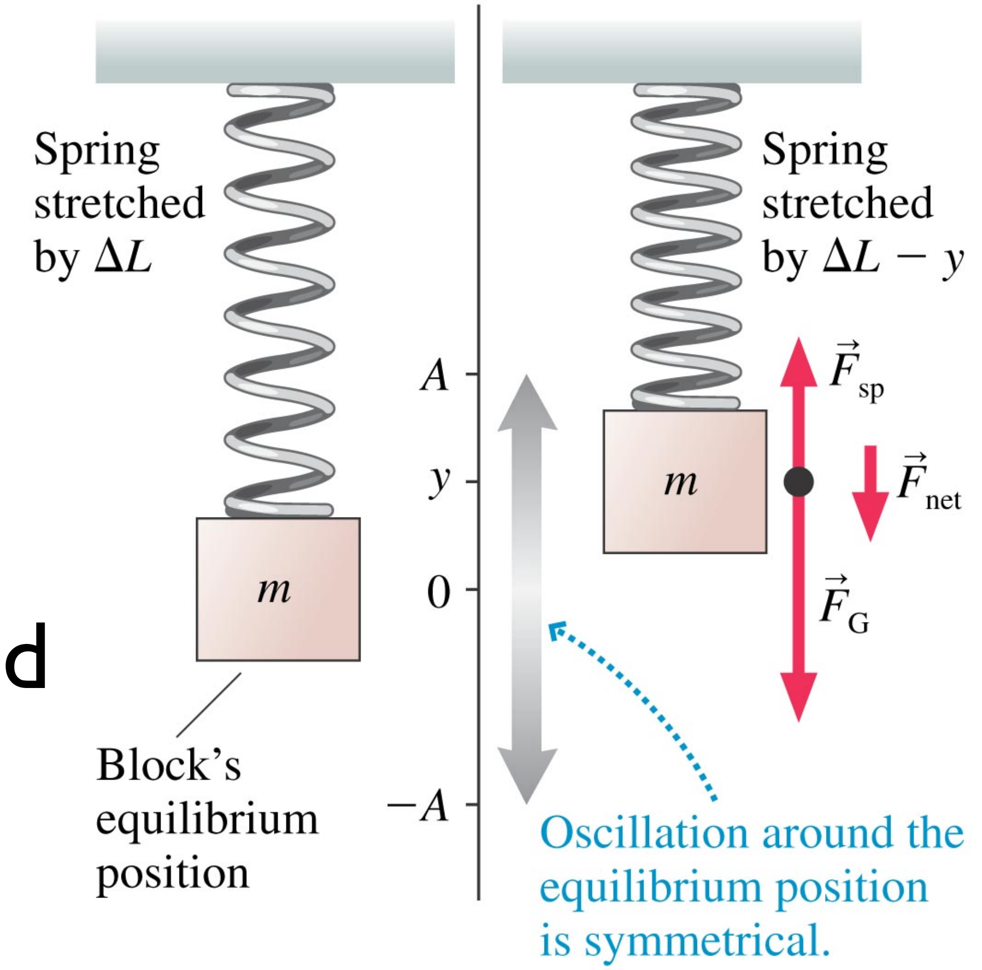
Hooke's law: $(F_{sp})_y = -k\Delta y = +k\Delta L$

Newton's law: $(F_{net})_y = (F_{sp})_y + (F_G)_y = k\Delta L - mg = 0$

$\Rightarrow \Delta L = \frac{mg}{k}$

Vertical Oscillations (II)

- Oscillation around equilibrium, y = 0
  (spring stretched)
  block moves upward, spring still stretched



$$(F_{net})_y = (F_{sp})_y + (F_G)_y = k(\Delta L - y) - mg$$

Using $k\Delta L - mg = 0$ (equilibrium), $(F_{net})_y = -ky$

- gravity ``disappeared"... ➡ as before: $y(t) = A\cos(\omega t + \phi_0)$

Example

- A 8 kg mass is attached to a spring and allowed to hang in the Earth's gravitational field. The spring stretches 2.4 cm. before it reaches its equilibrium position. If allowed to oscillate, what would be its frequency?

Pendulum (I)

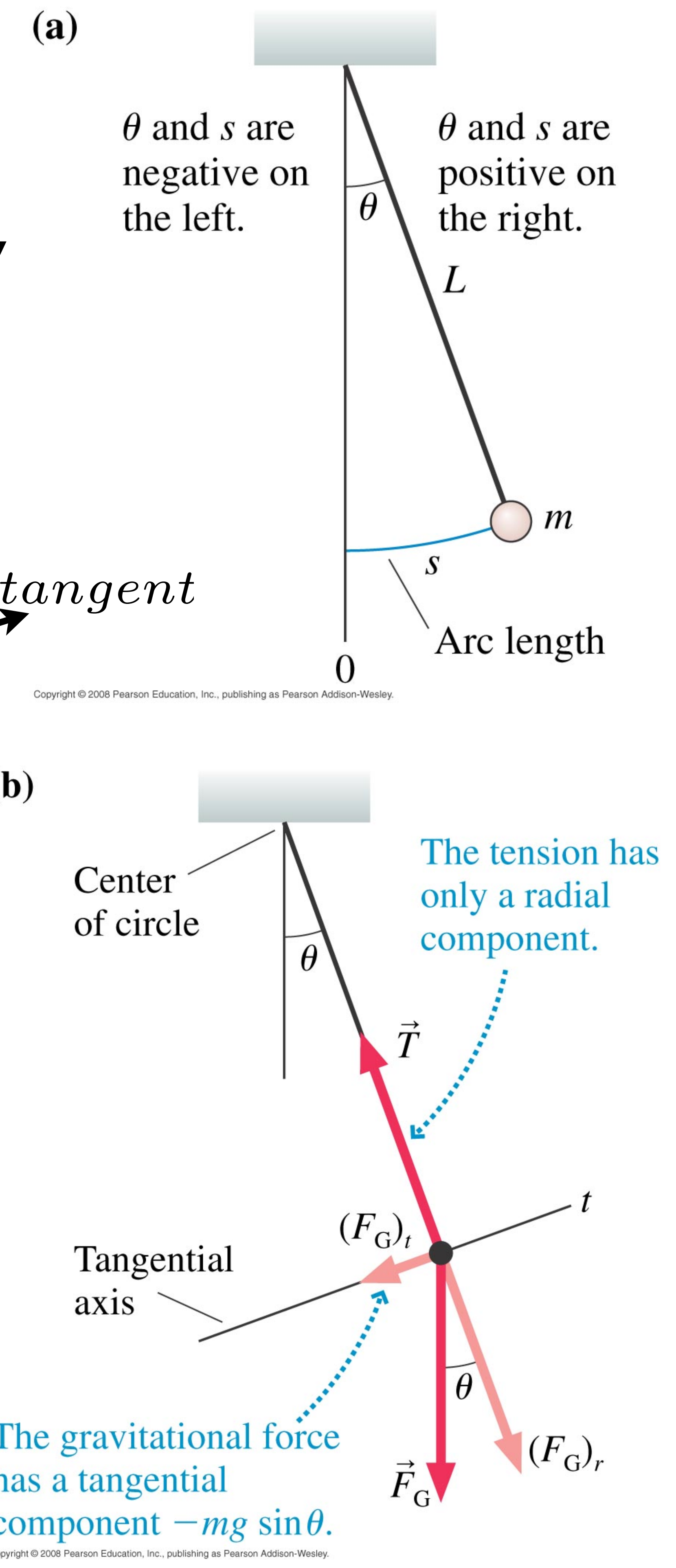
- Two forces: tension (along string) and gravity
- Divide into tangential and radial...

$$(F_{net})_{tangent} = (F_G)_{tangent} = -mg\sin\theta = ma_{tangent}$$

acceleration around circle

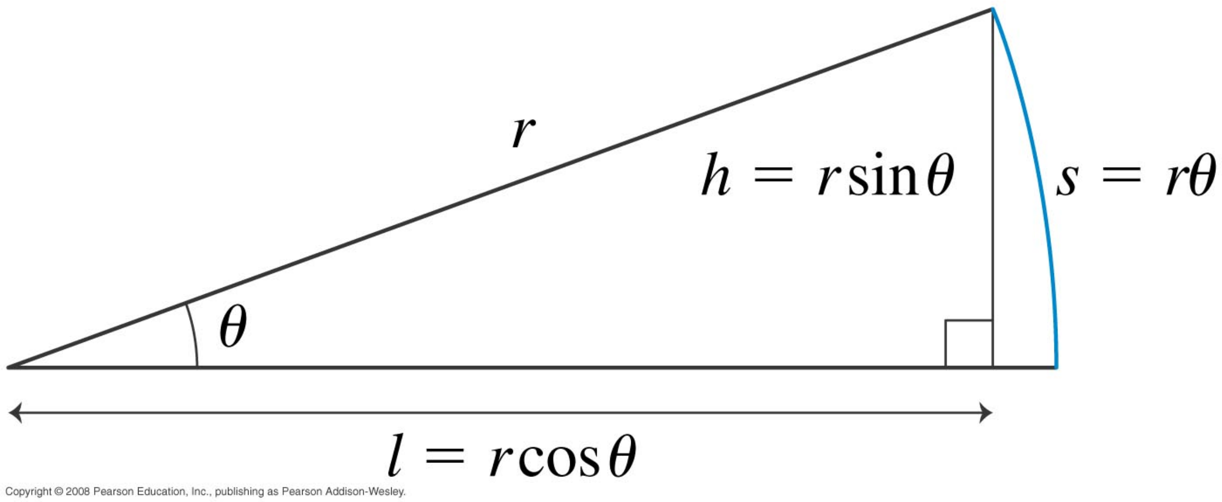
$$\frac{d^2 s}{dt^2} = -g\sin\theta$$

more complicated



Pendulum (II)

- Small-angle approximation



$$\sin\theta \approx \theta \ (\theta \text{ in radians})$$

$$(F_{net})_{tangent} \approx -\frac{mg}{L}s$$

$$\Rightarrow \frac{d^2s}{dt^2} = -\frac{g}{L}s$$ (same as mass on spring)

$$\Rightarrow s(t) = A\cos(\omega t + \phi_0) \text{ or } \theta(t) = \theta_{max}\cos(\omega t + \phi_0)$$

$$\omega = 2\pi f = \sqrt{\frac{g}{L}}$$

(independent of m, cf. spring)

Example

- The period of a simple pendulum on another planet is 1.67 s. What is the acceleration due to gravity on this planet? Assume that the length of the pendulum is 1m.

Summary

- linear restoring force ($\propto$ displacement from equilibrium)
  e.g. mass on spring, pendulum (for small angle)
- (x⟶y for vertical): $x(t) = A\cos(\omega t + \phi_0)$ $\quad v_x(t) = -\omega A \sin(\omega t + \phi_0)$
- $A,\ \phi_0$ determined by initial conditions (t=0)

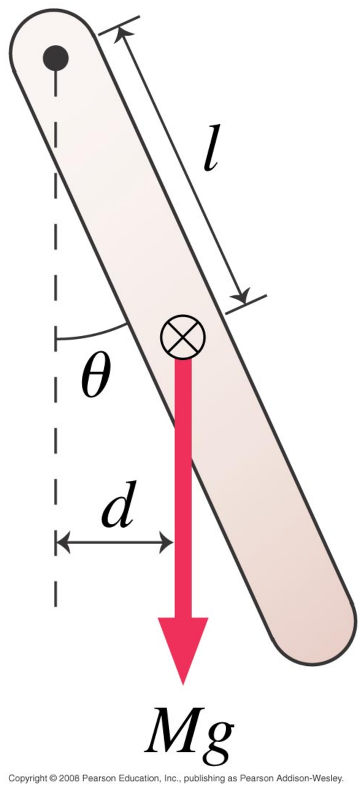
  $x_0 = A\cos\phi_0,\ v_{0\ x} = -\omega A \sin\phi_0$
- ω depends on physics ($\sqrt{k/m}$ or $\sqrt{g/L}$), not on $A,\ \phi_0$
- conservation of energy (similarly for pendulum):

$$1/2\ mv_x^2 + 1/2\ kx^2 = 1/2\ kA^2 = 1/2\ mv_{max}^2$$

KE PE turning point equilibrium

Pendulum (III)

- Physical pendulum (mass on string is simple pendulum)



(restoring) torque

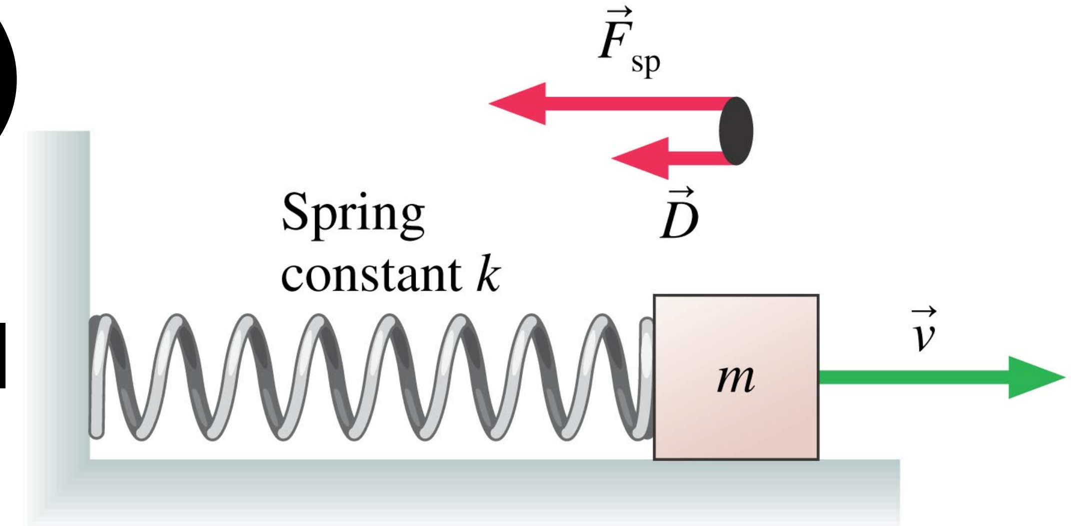
moment arm

$$\tau = -Mgd = -Mgl\sin\theta \approx -Mgl\theta \text{ (small angle)}$$

$$\alpha \text{ (angular acceleration)} = \frac{d^2\theta}{dt^2} = \frac{\tau}{I \text{ (moment of inertia)}} \Rightarrow \frac{d^2\theta}{dt^2} = \frac{-Mgl}{I}\theta$$

SHM equation of motion: $\omega = 2\pi f = \sqrt{\frac{Mgl}{I}}$

Damped Oscillations (I)



- dissipative forces transform mechanical into heat e.g. friction
- model of air resistance (b is damping coefficient, units: kg/s)

$$\bar{D} = -b\bar{v} \text{ (drag force)} \Rightarrow$$
$$(F_{net})_x = (F_{sp})_x + D_x = -kx - bv_x = ma_x$$
$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$ (equation of motion for damped oscillator)

- Check that solution is (reduces to earlier for b = 0)

$$x(t) = Ae^{-bt/2m}\cos(\omega t + \phi_0) \qquad \text{(damped oscillator)}$$

$$\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = \sqrt{\omega_0^2 - \frac{b^2}{4m^2}}$$

Damped Oscillations (II)

- Lightly damped: $b/(2m) \ll \omega_0$

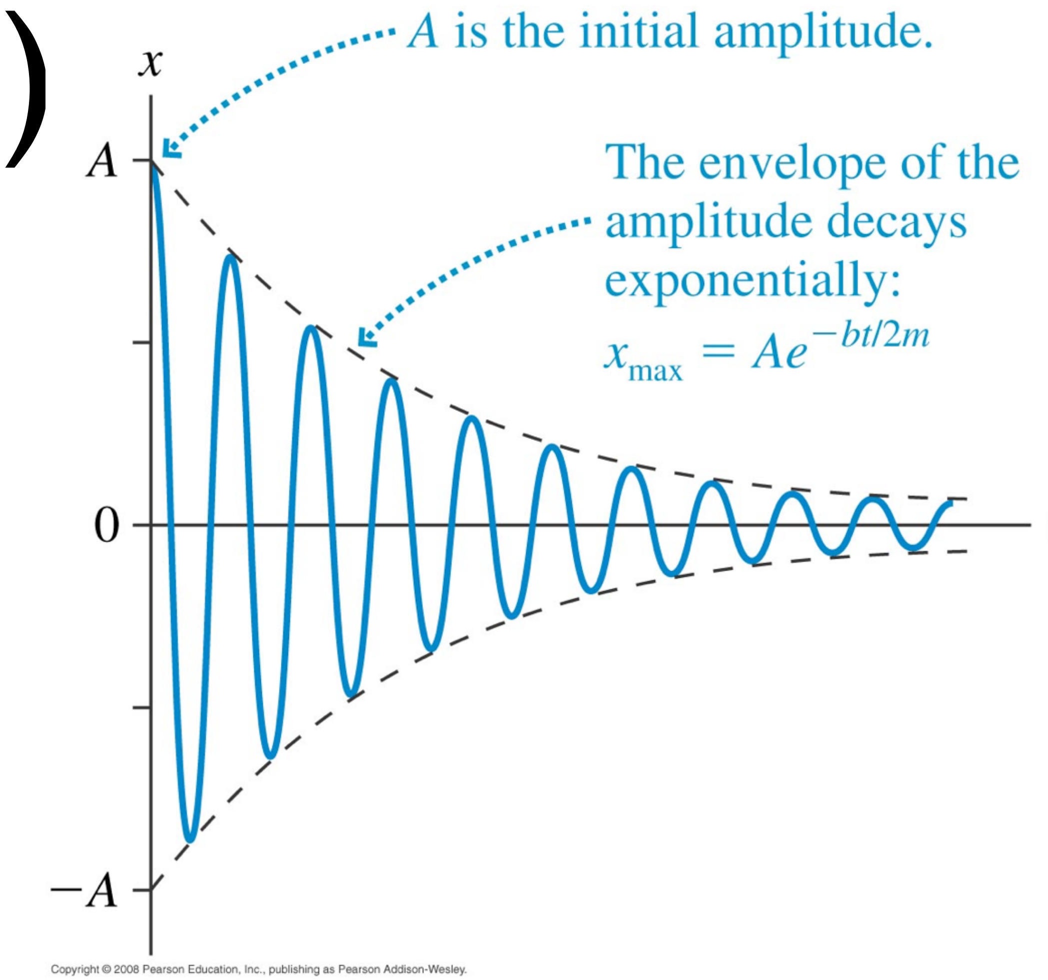
  $x_{max}(t) = Ae^{-bt/(2m)}$

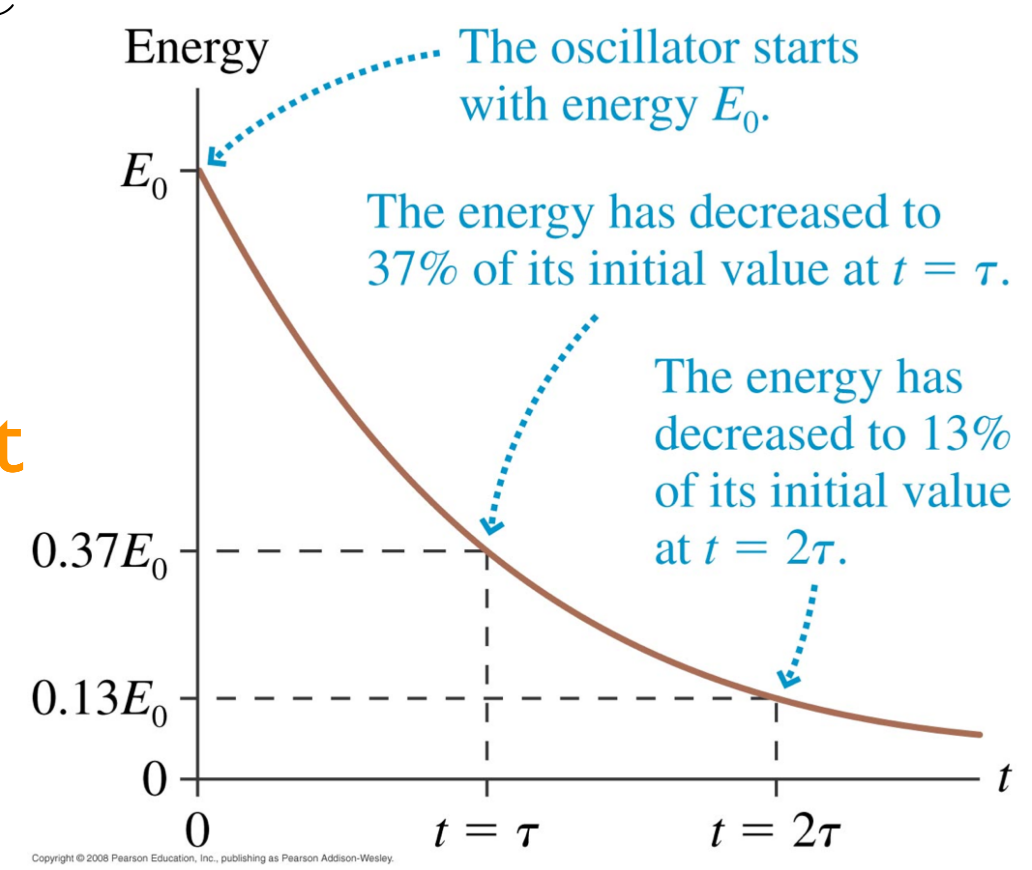
- Energy not conserved:

  $\tau \equiv \frac{m}{b}$ (time constant)

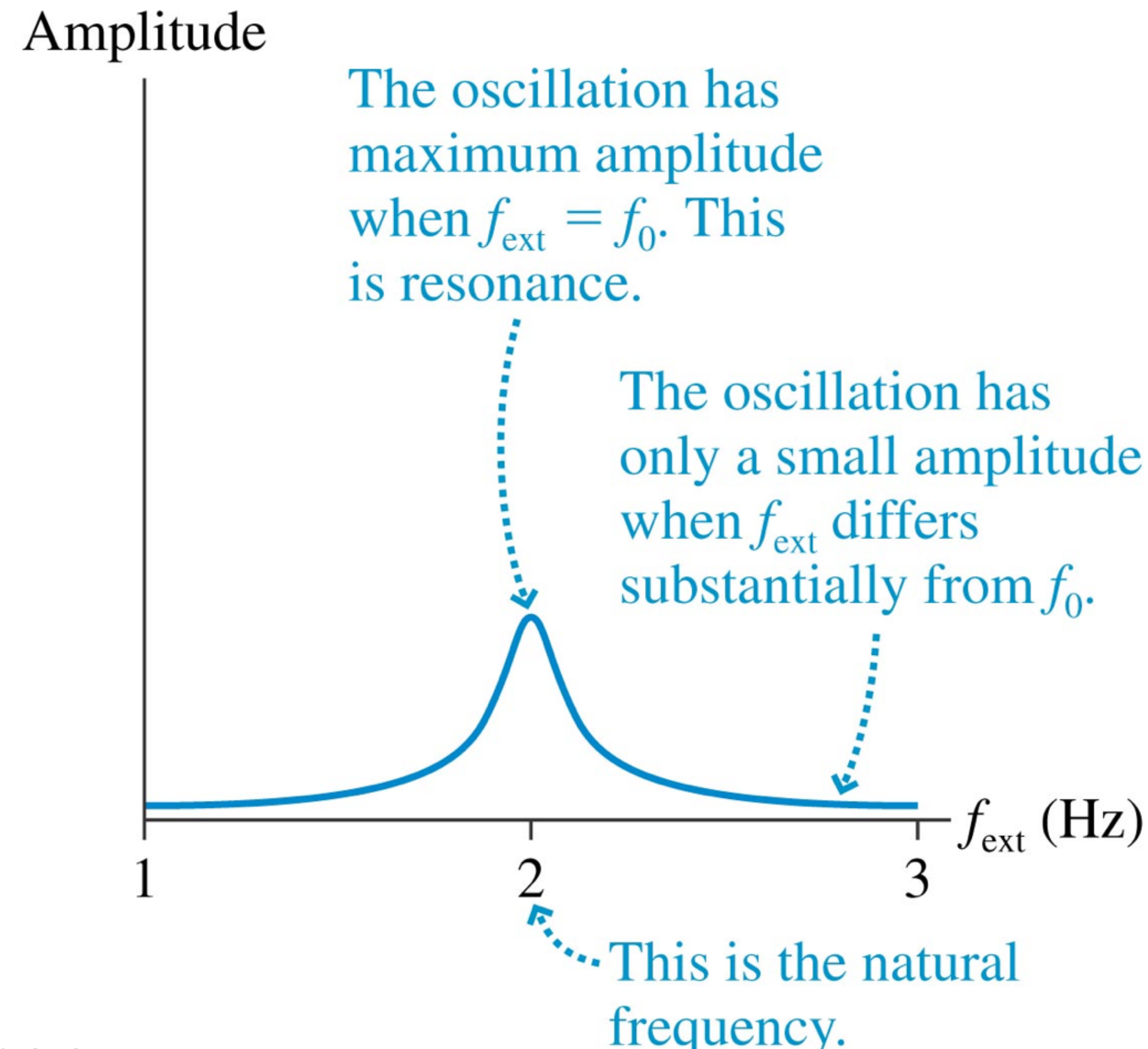
$$E(t) = \frac{1}{2}k\left(x_{max}\right)^2 = \left(\frac{1}{2}kA^2\right)e^{-t/\tau} = E_0 e^{-t/\tau}$$

- measures characteristic time of energy dissipation (or "lifetime"): oscillation not over in finite time, but "almost" over in time





Resonance



- Driven oscillations
  (cf. free with damping so far):
  periodic external force
  e.g. pushing on a swing

- f_0 : natural frequency of oscillation
  e.g. $\sqrt{k/m}$ or $\sqrt{g/L}$

- f_{ext}: driving frequency of external force

- amplitude rises as $f_{ext} \rightarrow f_0$: external forces pushes oscillator at same point in cycle, adding energy ($f_{ext} \neq f_0 \rightarrow$ sometimes add, other times remove, not in sync)

- amplitude very large: $f_{ext} = f_0 \;\; \text{(resonance)}$