Pendulum

Select LEARNING OBJECTIVES:

- Apply the sinusoidal equations of motion (position, velocity, acceleration) to SHO - i.e. fit them to the system using the initial conditions.
- Understand the system specific (pendulum) physical quantities.

TEXTBOOK CHAPTERS:

- Boxsand :: Springs and pendulums

WARM UP: Would the force below give rise to simple harmonic oscillation?

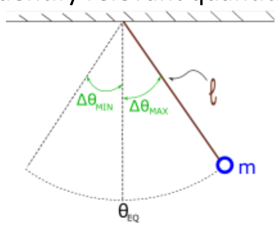
$$F_x = -7.5\,\Delta t$$

In this lecture we will begin to quantify the mass-spring system which is a case study of simple harmonic motion.

Assumptions

- No air resistance.
- String's mass is negligible.
- Small disturbance (i.e. small θ_{max}).

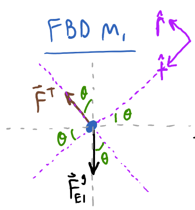
Below is a basic sketch to help identify relevant quantities for a pendulum system.



Equation of motion

Recall we can build equations of motion by applying Newton's 2nd law. To do this for the pendulum system, we will consider the mass displaced from its equilibrium location.

FBD m_1



$$\Sigma F_t = m_1 a_t$$

$a_t = r\alpha$ $\quad \omega = \frac{\Delta\theta}{\Delta t}$

$$F^g_{E1_t} = m_1 \ell \alpha$$

$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{\Delta}{\Delta t}\left(\frac{\Delta\theta}{\Delta t}\right) = \frac{\Delta^2\theta}{\Delta t}$

$$-m_1 g \sin\theta = m_1 \ell \frac{\Delta^2\theta}{\Delta t^2}$$

$$\frac{\Delta^2\theta}{\Delta t^2} = -\frac{g}{\ell}\sin\theta$$

FOR SMALL θ $\quad \sin\theta \approx \theta$

$$\boxed{\frac{d^2\theta}{dt^2} \approx -\frac{g}{\ell}\theta} \xleftarrow{\text{CALCULUS}} \boxed{\frac{\Delta^2\theta}{\Delta t^2} \approx -\frac{g}{\ell}\theta}$$

GENERAL SOLUTION

$$\boxed{\theta(t) = \pm\,\theta_{MAX} \begin{smallmatrix}\sin\\ \text{or}\\ \cos\end{smallmatrix}(\omega t)}$$

SLOPE

$$\Omega(t) = \pm\,\Omega_{MAX} \begin{smallmatrix}\cos\\ \text{or}\\ \sin\end{smallmatrix}(\omega t)$$

SLOPE

$$\alpha(t) = \pm\,\alpha_{MAX} \begin{smallmatrix}\sin\\ \text{or}\\ \cos\end{smallmatrix}(\omega t)$$

$+$ or $-$, sin or cos, θ_{max}: BASED OFF OF INITIAL CONDITIONS

- Mass oscillates back and forth between θ_{max} and θ_{min} with...
  - $|\theta_{max}| = |\theta_{min}|$
  - Angular frequency $\equiv \omega = \sqrt{\frac{g}{\ell}}$
  - Amplitude $\equiv A = |\theta_{max}|$
  - Period $\equiv T = 2\pi\sqrt{\frac{\ell}{g}}$
  - Frequency $\equiv f = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}}$
  - Max angular velocity $\equiv \Omega_{max} = \omega\,\theta_{max}$

- Max angular acceleration $\equiv \alpha_{max} = \omega\,\Omega_{max} = \omega^2\,\theta_{max}$
- Total energy $\equiv E_{tot} =$ KE + U^g = U^g_{max} = KE_{max}

$$E_{tot} = \frac{1}{2} m\,\ell^2\,\Omega_1^2 + \frac{1}{2} m\,g\,\ell\,\theta_1^2 = \frac{1}{2} m\,g\,\ell\,A^2 = \frac{1}{2} m\,\ell^2\,\Omega_{max}^2$$

PRACTICE: One person swings on a swing and finds the period is 3.0 seconds. Then a second person of equal mass joins him. With two people swinging, the period is

(a) 6.0 s
(b) > 3.0 s but not necessarily 6.0 s.
(c) 3.0 s
(d) < 3.0 s but not necessarily 1.5 s.
(e) 1.5 s

GENERAL: $\omega = \frac{2\pi}{T}$

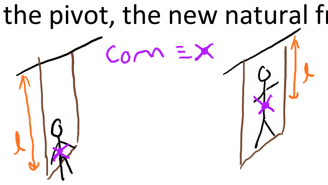
PENDULUM: $\omega = \sqrt{\frac{g}{\ell}}$

$\frac{2\pi}{T} = \sqrt{\frac{g}{\ell}}$

$T = 2\pi\sqrt{\frac{\ell}{g}}$ ... INDEPENDENT OF MASS

PRACTICE: A person swings on a swing. When the person sits still, the swing oscillates back and forth at its natural frequency. If, instead the person stands on the swing in such a way that her center of mass is always directly between her feet and the pivot, the new natural frequency of the swing is

(a) greater.
(b) the same.
(c) smaller.

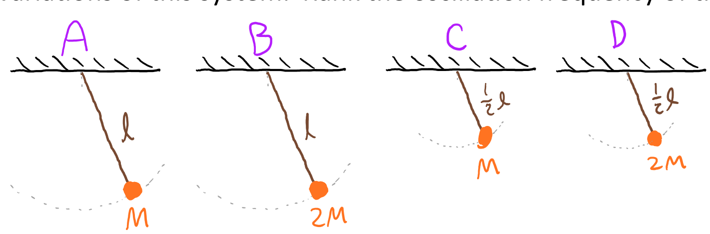


$\left.\begin{aligned} \omega &= 2\pi f \\ \omega &= \sqrt{\frac{g}{\ell}} \end{aligned}\right\}\ f = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}}$

IF $\ell\downarrow$

THEN $f\uparrow$

PRACTICE: The simple pendulum shown in case A consists of a mass M attached to a massless string of length L. If the mass is pulled to one side, a small disturbance, and released, it will swing back and forth. Cases B, C, and D are variations of this system. Rank the oscillation frequency of the masses.



$f = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}}$

$f_A = f_B < f_C = f_D$

PRACTICE: Each group should have a piece of string and a mass. Use *only* these two items to estimate g, the acceleration due to gravity in units of m/s^2.

$\omega = \sqrt{\frac{g}{\ell}}$

$\omega^2 = \frac{g}{\ell}$

$g = \ell\omega^2$

$g = \frac{4\pi^2}{T^2}\ell$

MEASURE

PRACTICE: A simple pendulum of mass 0.25 kg undergoes simple harmonic oscillation as described by the equation below. Determine the…

$\theta(t) = \frac{\pi}{18}\cos\left(\frac{\pi}{2}t\right)$

(a) …amplitude. $\frac{\pi}{18}$ rad or 10°
(b) …period. 4 sec
(c) …frequency. 0.25 Hz
(d) …maximum angular speed that the oscillator reaches. 0.274 $\frac{\text{rad}}{\text{s}}$
(e) …maximum angular acceleration of the oscillator. 0.431 $\frac{\text{rad}}{\text{s}^2}$
(f) …length of the pendulum. 3.97 m
(g) …total energy. 0.148 J

… general …

$\theta(t) = \theta_{max}\cos(\omega t)$

$\omega = \frac{\pi}{2}$

$2\pi f = \frac{\pi}{2}$

$f = \frac{1}{4}$ Hz $\quad T = 4$ sec

$\Omega_{max} = \omega\theta_{max}$
$= \frac{\pi}{2}\left(\frac{\pi}{18}\right) \approx 0.274 \frac{\text{rad}}{\text{s}}$

$\alpha_{max} = \omega^2\theta_{max}$
$= \left(\frac{\pi}{2}\right)^2\left(\frac{\pi}{18}\right) \approx 0.431 \frac{\text{rad}}{\text{s}^2}$

$T = 2\pi\sqrt{\frac{\ell}{g}}$

$\ell = \frac{T^2 g}{4\pi^2} \approx 3.97$ m

$E_{tot} = KE_{max} = U_{g\,max}$
$= \frac{1}{2}m\ell^2\Omega_{max}^2 = \frac{1}{2}mg\ell A^2$
≈ 0.148 J

QUESTIONS FOR DISCUSSION:

(1) You are preparing for a mission to a distant unknown planet. Unfortunate you only have room left for either a mass-spring oscillator or a pendulum oscillator. You intend to determine the gravitational constant of this planet via multiple methods. Which one would you pack system do you pack and why?
(2) A grandfathers clock is "losing" time because its pendulum moves too slowly. Assume that the pendulum is a massive bob at the end of a string. The motion of this pendulum can be sped up by (list all that work):
   a. Shortening the string.
   b. Lengthening the string.
   c. Increasing the mass of the bob.
   d. Decreasing the mass of the bob.