Simple harmonic oscillators

Select LEARNING OBJECTIVES:

- Be able to identify the features of a system that oscillates - i.e. systems with a restoring force and a potential energy well.
- Be able to identify if an oscillation will be Simple Harmonic - i.e. systems with a linear restoring force and a quadratic potential energy well.
- Apply the sinusoidal equations of motion (position, velocity, acceleration) to SHO - i.e. fit them to the system using the initial conditions.
- Understand the universal physical quantities of a SHO.

TEXTBOOK CHAPTERS:

- Boxsand :: Simple harmonic oscillators

WARM UP: Examples of oscillations?

Let's summarize what we have studied so far:

- **Newtonian mechanics**
  - Concepts encountered
    - position, velocity, acceleration
    - kinematics
    - Newton's laws of motion
    - ***Conservation of momentum.***
    - ***Conservation of mechanical energy*** (translational and vibrational kinetic energy, gravitational potential energy, spring potential energy)
    - Statics
  - Summary: How does a mechanical system evolve in time? Position and velocity are variables which describe the state of the system. The system's evolution governed first by Newton's laws of motion, then later by a more elegant way, ***conservation laws***.
    - Example: Object bouncing on a trampoline (system = object + trampoline + earth). The object had a state (position and velocity) which changed as a function of time which we were able to predict using ***conservation of energy***.
- **Thermodynamics**
  - Concepts encountered
    - kinetic theory of gases
    - thermodynamic equilibrium
    - equations of states
    - ***Conservation of energy*** (1^{st} law of thermodynamics)
    - calorimetry (application of ***conservation of energy***)
    - thermodynamic processes and cycles
  - Summary: How does a thermodynamic system evolve from one equilibrium state to another? Position and velocity were abandoned as state variables because there were too many particles (~10^{24}); instead we used thermodynamic state variables (T, P, V, N, E^{th}) which are macroscopic descriptions rooted in very microscopic ideas. Using ***conservation laws*** we were able to predict future equilibrium states of a system (defined by thermodynamic state variables) after going through various thermodynamic processes and cycles.
    - Example: A gas enclosed by a freely movable piston. The gas had an equilibrium state (T, P, N, V, E^{th}) which changed to a new equilibrium state if the piston compressed the gas. We were able to predict the new equilibrium state of a gas by applying ***conservation of energy*** (1st law of

thermodynamics) to an ideal gas model of an equation of state.

- **Fluid mechanics**
  - Concepts encountered
    - hydrostatics (pressure at a depth)
    - ***conservation of mass flow rate*** (continuity equation)
    - ***conservation of energy density*** (Bernoulli's equation)
  - Summary: How does a fluid mechanical system evolve if parameters of the confining enclosure are altered? Speed, pressure, and height are the relevant variables needed to define a state for our simplified fluid mechanical studies. After making simplifying assumptions (defining an ideal fluid) we were able to use ***conservation of mass flow rate*** and ***conservation of energy*** to arrive at two equations: continuity equation, and Bernoulli's equation. Bernoulli's equation simplifies to the hydrostatics pressure at a depth analysis first studied when we encountered fluid mechanics.
    - Example: Water is moving through a garden hose at some initial velocity. If a person places their finger partially over the opening of the hose we were able to predict the new velocity at this location using ***conservation of energy*** (Bernoulli's equation) and ***conservation of mass flow rate*** (continuity equation).

By going over this quick recap of what we have done so far, it is hopefully evident that conservation laws help us analyze a wide range of physical phenomenon. In fact, we are still not done yet. We will apply conservation laws to more phenomenon; in ph213 we will explore conservation of charge and expand on our conservation of energy model to incorporate electric potential energy to help analyze motion of charged particles. My hope is that no matter where you wind up in life, you can take a quick moment to appreciate the world around you and how conservation laws help us analyze what you are observing.

So conservation laws help us analyze all types of phenomenon, but there is one type of phenomenon that is perhaps the most prevalent type of phenomena in our universe, oscillations. Just as you have come to appreciate how conservation laws help us analyze phenomenon, I also hope you will eventually appreciate that oscillations occur all around you.

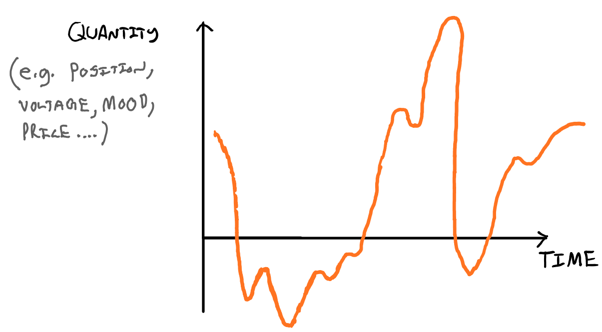
Oscillations

Oscillations are any phenomenon which contain a repetitive quantity.

Examples

- Heat beat
- Stock market data
- Emotional state (mood)
- Mass connected to a spring (e.g. car's suspension)
- Electronics (e.g. LC circuits)
- Quantum oscillators (e.g. binding of atoms)
- Celestial motion (e.g. moon orbits Earth, Earth orbits sun, solar system orbits black hole at center of galaxy…)

Note that oscillations are pervasive on all scales from microscopic to macroscopic. Below is a plot of any general oscillation, depicting the quantitate that is repeating vs time.



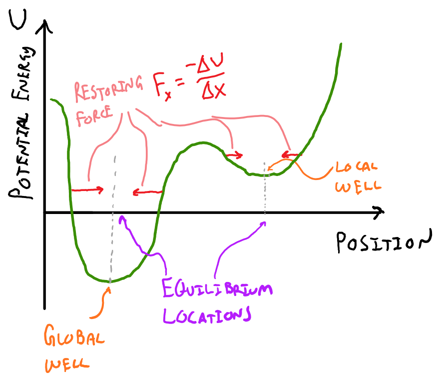
The above graphical example is quite complex, oscillations can also be simple (i.e. sinusoidal) as well. Before we can explore the more complex oscillations we must first tackle simple oscillatory motion. But before we do, let's define some common features of general oscillations.

Features of general oscillations

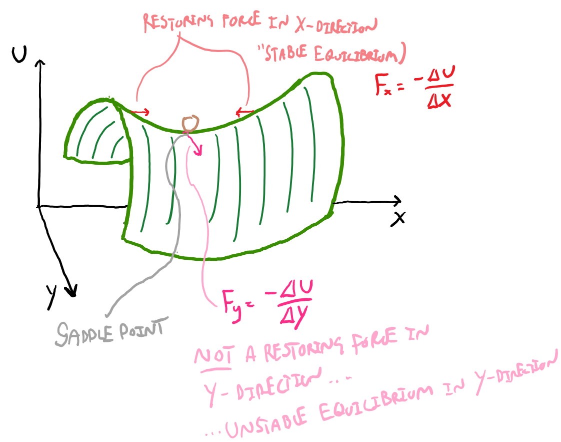
All oscillations share a few common features as described below.

- *Equilibrium location:* Sate where quantity is constant (not changing). Stable equilibrium locations are the common feature of oscillators. Do not confuse stable equilibrium with unstable equilibrium locations.
- *Restoring force*: When a system is displaced from its equilibrium state, there is a force that drives it back to that equilibrium state. This force is referred to as the restoring force.
- *Potential energy well:* The potential energy function of the system (i.e. U) must have a local or global minimum value where the potential energy increases if a small displacement from this location occurs. Two examples are shown below.

1-D (EXAMPLE... BALL ON A METAL RAMP)



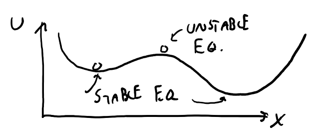
2-D (EXAMPLE... m ← m → m)



- *Disturbance from equilibrium:* Initial interaction that causes system to depart from equilibrium state.

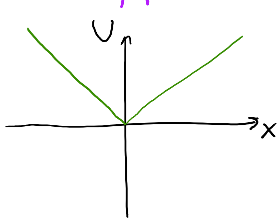
PRACTICE: Which of the following is/are necessary to make an object oscillate?

(a) A stable equilibrium
(b) An unstable equilibrium
(c) Little or no friction
(d) A disturbance

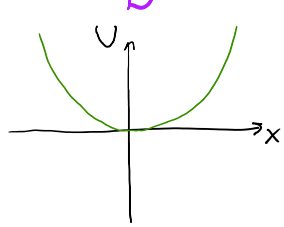


PRACTICE: Which of the following graphs of potential energy vs position can NOT give rise to oscillatory motion?

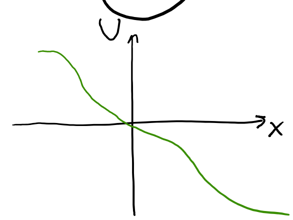
A



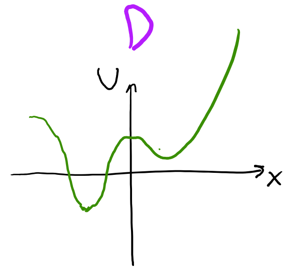
B



C

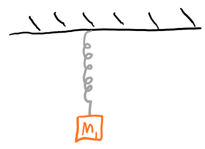


D



PRACTICE: An object hangs motionless from a spring. When the object is pulled down, the sum of the elastic potential energy and the gravitational potential energy of the object-spring-Earth system

(a) increases.
(b) stays the same.
(c) decreases.



$\cancel{\Delta KE}^{0} + \Delta U^e + \Delta U^g = W_{EXT} + \cancel{Q}^{0}$

IF DONE SLOWLY $KE \approx 0$

SO...

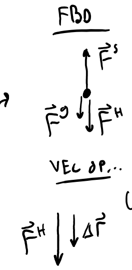
$\Delta U^e + \Delta U^g = W_{HAND}$

$+\frac{1}{2}k\Delta x^2 + (-mg\Delta y) = W_{HAND}$

WHICH IS BIGGER?

LOOK AT W_{HAND}

FBD



$W_{HAND} = \vec{F}^H \cdot \Delta\vec{r}$

$= |\vec{F}^H||\Delta\vec{r}|\cancel{\cos\theta}^{+1}$

$W_{HAND} = (+)$

*ENERGY OF OSCILLATION COMES FROM WHAT EVER DID WORK TO DISPLACE IT INITIALLY.

Simple harmonic oscillation (SHO)

As it turns out, complex oscillations can be approximated by a summation of simpler sinusoidal oscillations; a concept we will visit in the near future. Thus in the next few lectures we will focus our attention on various examples of simple harmonic oscillators to help ourselves become familiar with oscillatory motion. Note that simple harmonic oscillations share the same features listed above in the general features section; what differentiates a harmonic oscillator is the functional form of the restoring force and the potential well. These SHO specific features are listed below.

Features of SHOs

- *Restoring force:* Restoring force is linearly dependent on the displacement from equilibrium ($\Delta x = x - x_{eq}$). Another way to state this is that the restoring force is directly proportional to the displacement from equilibrium.

$$F_x^{RESTORE} \propto -\Delta x$$

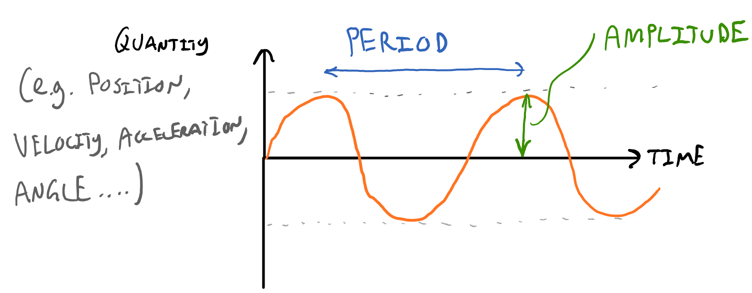
- *Potential energy well:* The potential energy function that arises the SHO restoring force is quadratic with respect to the displacement from equilibrium. Another way to state this is that the potential energy function is directly proportional to the square of the displacement from equilibrium.

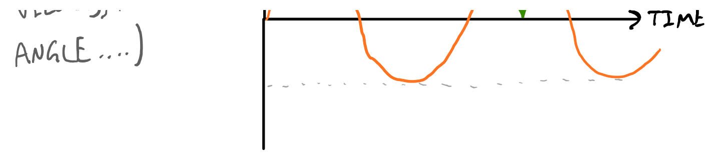
$$U \propto \Delta x^2$$

Recall from ph211 that we can recover forces from potential energy functions by looking at "3-D slops" (gradients). Thus the restoring force and the potential energy are actually dependent on each other; basically because the restoring force is linearly dependent on displacement the potential energy function is quadratic (and visa versa).

- *Amplitude (A):* The measure of how far away a quantity deviates from its equilibrium location.

- *Period (T):* The amount of time it takes a quantity to complete one full cycle. The period of any simple harmonic oscillator is independent of the amplitude of oscillation. If the system is losing energy the amplitude of oscillation will decrease, the period will slightly shift to a new value but will remain a constant independent of amplitude.





- *Frequency (f):* The number of cycles that a quantity has completed in a given unit of time. SI units are 1/s often referred to has Hertz (Hz). Thus the period and frequency are related to each other via the relationship below…

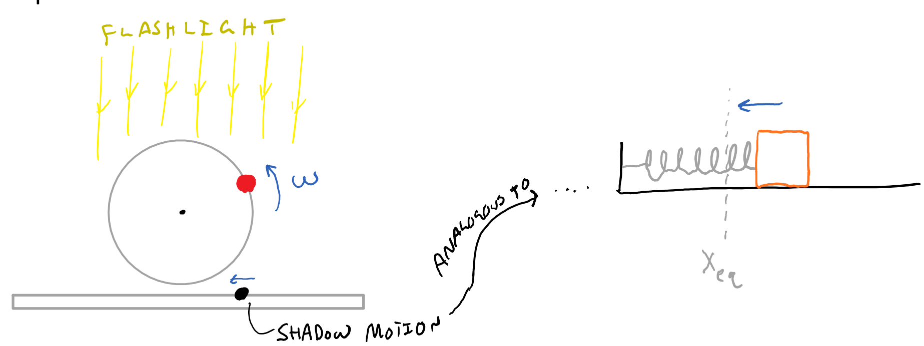
$$\boxed{T = \frac{1}{f}}$$

- *Angular frequency (ω):* Simple harmonic oscillation is analogous to uniform circular motion, thus we use angular quantities to describe SHO. SI units are radians/second.

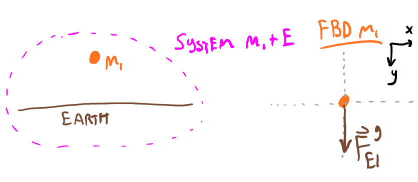
$$\boxed{\omega = 2\pi f = \frac{2\pi}{T}}$$

  - Connection to uniform circular motion (UCM)

    Consider a red ball connected to a thin circular pipe bent into a circle which is rotating at a constant angular velocity above a level surface. Now take a flashlight and shine it vertically downwards onto the rotating contraption and observe the shadow of the red ball on the level surface below. You will notice that the back and forth motion of the shadow is connected to the rotational motion of the red ball. So even though most of our simple harmonic oscillators are not moving in a circle (mass on spring, pendulum…), it is natural to use uniform circular motion terminology since simple harmonic oscillator quantities repeat themselves with a constant period.



- *Equation of motion:* Equations of motion are not specific to SHO, however I feel this is a good place to introduce the concept. Recall that most of what we do in physics is trying to determine the evolution of a system. A nice compact way to describe the evolution of a system is to use an equation of motion, which is just a mathematical relationship between position, velocity, and acceleration. From the equation of motion you can construct position, velocity, and acceleration functions of time. You actually already have experiences with equations of motion. Every time you applied Newton's second law to a system and simplified it down to find an acceleration, really what you were doing was constructing an equation of motion. Below is an example of constructing the equation of motion for a non-SHO that you have studied in ph211 (an object falling near the surface of the earth with no air resistance).



$$\Sigma F_y = m_1 a_y$$
$$|\vec{F}^g_{E1}| = m_1 a_y$$
$$m_1 g = m_1 a_y$$
$$g = a_y$$

$$\Delta V_y = \frac{\Delta y}{\Delta t}$$

$$a_y = \frac{\Delta V_y}{\Delta t} = \frac{\Delta}{\Delta t}\left(\frac{\Delta y}{\Delta t}\right) = \frac{\Delta(\Delta y)}{\Delta t^2} = \frac{\Delta^2 y}{\Delta t^2}$$

$m_1 g = m_1 a_y$

$$g = a_y \xleftarrow{\text{CALCULUS}} \left\{ \begin{array}{l} g = a_y \\ g = \dfrac{\Delta^2 y}{\Delta t^2} \end{array} \right.$$

$$g = \frac{d^2y}{dt^2}$$

$$a_y = \frac{\Delta V_y}{\Delta t} = \frac{\Delta}{\Delta t}\left(\frac{\Delta y}{\Delta t}\right) = \frac{\Delta(\Delta y)}{\Delta t^2} = \frac{\Delta^2 y}{\Delta t^2}$$

"SLOPE OF A SLOPE"

FROM THIS EQUATION OF MOTION WE CAN GET ... $a_y(t)$, $V_y(t)$, $y(t)$

Not too bad right? Why go through all this trouble though? Well, equations of motion are compact mathematical relationships that allow us to quickly identify the expected type of motion for a system. They also give quick insight when comparing different physical system's with similar equations of motion. For example, the general equation of motion for a SHO is shown below. Thus if we ever construct an equation of motion for a system that looks similar to the SHO equation of motion below, we know exactly what type of behavior to expect from the get go.

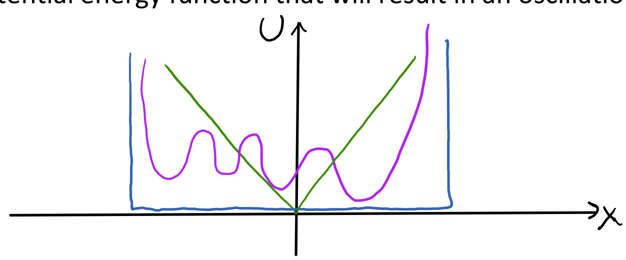
- General SHO equation of motion

$$\boxed{\frac{\Delta^2 X(t)}{\Delta t^2} = -\omega^2 X(t)} \qquad \boxed{\frac{d^2x}{dt^2} = -\omega^2 X(t)} \xrightarrow{\text{GENERAL SOLUTION}} \boxed{\begin{array}{c} X(t) = \pm X_{MAX} \sin(\omega t) \\ \text{OR} \\ X(t) = \pm X_{MAX} \cos(\omega t) \end{array}}$$

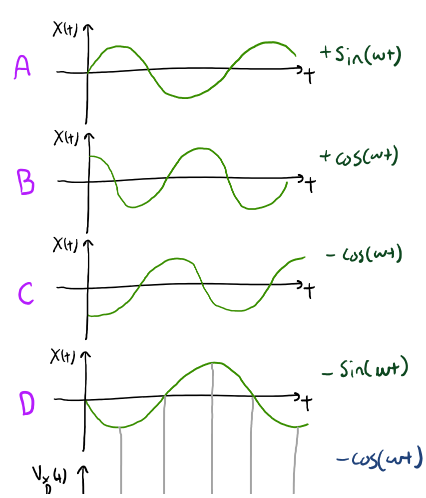
AMPLITUDE (X_{MAX}), + OR −, SIN OR COS } DEPENDS ON INITIAL CONDITIONS

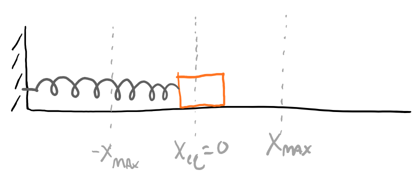
What does this equation of motion tell us? Well, if you take some function x(t) and take the slope of it twice, you will get the same function x(t) back but scaled by a constant (- ω^2). The only type of function that has this behavior is sinusodial functions. Thus for SHO, the position as a function of time can be described by a sine or cosine function. Likewise the velocity and acceleration functions will also be sine and cosine functions. The details of the sine or cosine function depend on what are called initial conditions which correspond to the state of the system at time equals zero (t = 0).

PRACTICE: Sketch a potential energy function that will result in an oscillation that is not simple harmonic.



PRACTICE: A mass on a spring undergoes simple harmonic oscillation. At t=0 s the mass passes through the equilibrium point moving to the right (+ x direction). Which of the following plots could represent the object's position as a function of time?

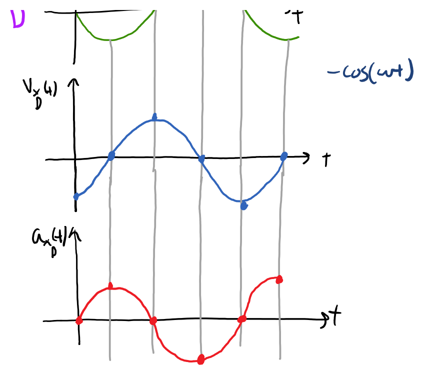




Is the velocity at t=0 positive, negative, or zero for each case? VELOCITY IS SLOPE OF POSITION VS TIME

V_A	V_B	V_C	V_D
+	0	0	−

Is the acceleration at t=0 positive, negative, or zero for each case? ACCELERATION IS SLOPE OF VELOCITY VS TIME ... OR CURVATURE OF POSITION VS TIME



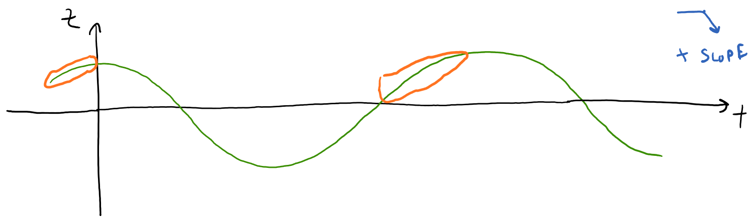
Is the acceleration at t=0 positive, negative, or zero for each case?

ACCELERATION IS SLOPE OF VELOCITY vs TIME
... OR CURVATURE OF POSITION VS TIME

a_A	a_B	a_C	a_D
0	-	+	0

... OR LOOK AT $\vec{F}^s \propto \vec{a}$

PRACTICE: An object suspended from a spring is undergoing simple harmonic motion up and down as indicated by the graph below. Select a point on the graph where v_z is positive and a_z is negative.

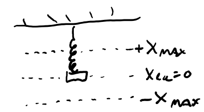


+ SLOPE

CURVE DOWN

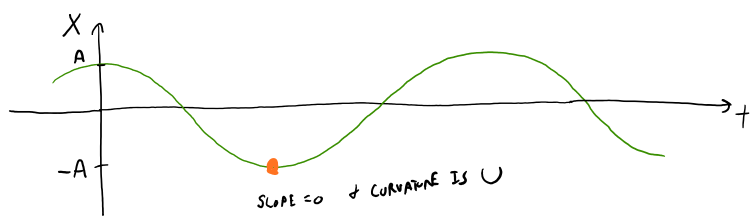
... OR... CONSTRUCT $V_z(t)$ AND $a_z(t)$ PLOTS

... OR... LOOK AT $\vec{F}^s$

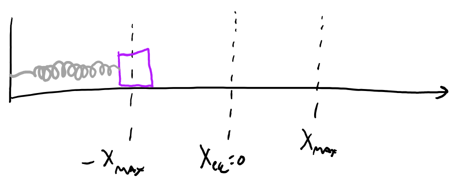


PRACTICE: Below is a position vs time graph for a mass on a spring undergoing simple harmonic oscillation with the positive direction to the right. What can you say about the velocity and the force at the instant indicated by the orange dot?

(a) Velocity is zero; force is to the right.
(b) Velocity is zero; force is to the left.
(c) Velocity is negative; force is to the right.
(d) Velocity is negative; force is to the left.
(e) Velocity is positive; force is to the right.
(f) Velocity is positive; force is to the left.



SLOPE = 0 & CURVATURE IS ∪



QUESTIONS FOR DISCUSSION:

(1) For a simple harmonic oscillator, when (if ever) are the displacement and velocity vectors in the same direction?
(2) For a simple harmonic oscillator, when (if ever) are the displacement and acceleration vectors in the same direction?