Angular momentum

Select LEARNING OBJECTIVES:

i. Introduce the definition of angular momentum.
ii. Understand when angular momentum is conserved (i.e. no net external forces acting on the system).
iii. Demonstrate the ability to properly define a system and apply conservation of angular momentum.
iv. Demonstrate the ability to analyze a torque vs time graph, relating area to change in angular momentum.
v. Be able to calculate the angular momentum of a system when given moment of inertia and angular velocity
vi. Be able to calculate a change in angular velocity if moment of inertia changes (i.e. a figure skater pulls in his arms, changing his moment of inertia. Since angular momentum is conserved, his angular velocity must change to compensate the change in moment of inertia).
vii. Understand that angular momentum of a system changes if the net external torque on the system is non-zero.

TEXTBOOK CHAPTERS:

- Giancoli (Physics Principles with Applications 7th) :: N/A
- Knight :: 12.4
- BoxSand :: Rotational Mechanics (Angular momentum)

WARM UP: What do you remember about conservation of momentum and or the momentum-impulse theorem?

When we first introduced velocity, momentum, force, etc… we worked under a model that assumed all objects are point particles and the resulting motion equations we derived/encountered were all describing the motion of the center of mass of the object or system. Our study of rotational mechanics started off by expanding our model and considering the consequences of taking into account the location that forces acted on an object rather than assuming the forces all act through the center of mass of an object or system. This added complexity didn't nullify any of our previous efforts; all laws and conservation statements from before are still valid and must hold true. In fact, you have seen so far that using both the old translational point particle model and the new rotational rigid body model in tandem when analyzing problems is often beneficial. Recall back to our first introduction to momentum when we scaled the translational velocity of an object by its mass. Mathematically, this momentum is described below:

$$\vec{p} = m\vec{v}$$

I will refer to this as translational momentum, since it is the momentum of an object or system assuming that all of the mass is concentrated at the center of mass, thus it can't rotate about it's center; it can only translate (i.e. move up down left or right). However, we can define an angular momentum mathematically as:

$$\boxed{\vec{L}_o = \vec{r}_o \times \vec{p}} \qquad \text{OR} \qquad \vec{L}_o = \vec{r}_o \times m\vec{v}$$

ANGULAR MOMENTUM ABOUT SOME REFERENCE AXIS "O"

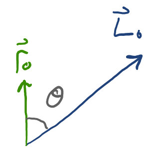
POSITION VECTOR POINTING FROM REFERENCE AXIS TO THE OBJECT

TRANSLATIONAL MOMENTUM OF OBJECT

Why that mathematical form? I won't go into any rigorous explanations why, just a simple analogy argument. Recall that torque is also a cross product of some moment arm with an applied force which results in rotational behavior. Since we want an angular momentum (i.e. momentum related to rotation) we can conclude that the cross product is necessary to account for the rotational (i.e. angular) part, so it seems natural to take some position vector from a reference axis and cross it with our definition of translational momentum. Our rules for dealing with cross products still hold true. Likewise, we will ignore the vector nature of angular momentum, and only deal with its magnitude and a positive or negative if rotating clockwise or counterclockwise.

$$|\vec{L}_o| = |\vec{r}_o||\vec{p}|\sin\theta$$

SMALLEST ANGLE BETWEEN $\vec{r}_o$ AND $\vec{p}$ WHEN TAIL TO TAIL

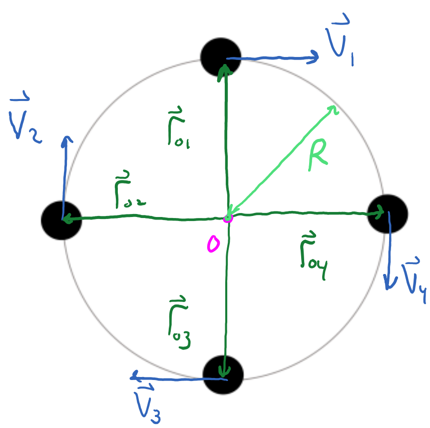


$$\boxed{L_o = \pm |\vec{r}_o| m |\vec{v}| \sin\theta}$$

ADD + OR − IN AD HOC BASED ON DIRECTION OF ROTATION .... CCW(+)
CW(−)

PRACTICE: Four point particles each of mass m are fixed to a negligible mass wire bent into a circle of radius R as shown below. If the masses are spinning clockwise around the center at a constant 60 rpm, what is the angular momentum of the 4 mass system?

1. 0 m R^2
2. - 25.1 m R^2
3. 60 m R^2
4. 240 m R^2
5. - 240 m R^2
6. - 1508 m R^2



$$\vec{L}_o = \vec{r}_o \times m\vec{V}$$

$$L_o = |\vec{r}_o| m |\vec{v}| \sin\theta$$

$$\Sigma L_o = -(L_{o,1} + L_{o_2} + L_{o_3} + L_{o_4})$$

W/ M, $|\vec{V}|$, $|\vec{r}|$, AND θ ALL THE SAME

$$\Sigma L_o = -4 L_{o,1}$$

$$= -4 |\vec{r}_{o_1}| m |\vec{v}| \cancel{\sin\theta_1}^{\,1}$$

$$= -4 R m \left(\frac{2\pi R}{T}\right)$$

$$\Sigma L_o = -8\pi m R^2 f$$

$$f = 60 \frac{\text{REV}}{\text{MIN}} \times \frac{1\,\text{MIN}}{60\,\text{SEC}} = 1 \frac{\text{REV}}{\text{SEC}}$$

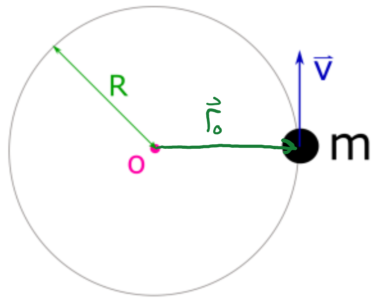
ALTERNATIVE

$$\Sigma L_o = -4 L_{o,1}$$

$$= -4 I_{o,1} \omega$$

$$= -4 (mR^2)(2\pi f)$$

It turns out that we can write angular momentum in a more compact form for objects rotating in a circle. First consider a single point particle of mass m rotating at a constant speed and radius around a point o as shown below.



$$\vec{L}_o = \vec{r}_o \times \vec{p}$$

$$\vec{L}_o = \vec{r}_o \times m\vec{V}$$

$$L_o = +|\vec{r}_o|\, m\, |\vec{V}|$$

$$L_o = Rm|\vec{V}| \qquad |\vec{V}| = \frac{\text{DIST}}{\text{TIME}} = \frac{\text{CIRCUMFERENCE}}{\text{PERIOD}}$$

$$L_o = Rm\left(\frac{2\pi R}{T}\right) \qquad T = \frac{1}{f}$$

$$L_o = mR^2\, 2\pi f \qquad 2\pi f = \omega$$

$$L_o = mR^2 \omega \qquad mR^2 = I_o \text{ FOR POINT PARTICLE}$$

$$L_o = I_o\, \omega$$

Follow the derivation carefully and you see that the angular momentum reduces to the moment of inertia for the point particle multiplied by the angular velocity of the point particle as it travels in a circle around o. We can generalize this simple scenario and claim that the moment of inertia for any shape rigid body is the moment of inertia for that rigid body multiplied by the angular velocity of the rigidi body.

$$\boxed{L_o = \pm I_o\, \omega}$$

ANGULAR VELOCITY

ANGULAR MOMENTUM ABOUT REFERENCE AXIS "O"

ADD + OR − AD HOC BASED ON CCW(+) CW(−)

MOMENT OF INERTIA FOR OBJECT

IN GENERAL...

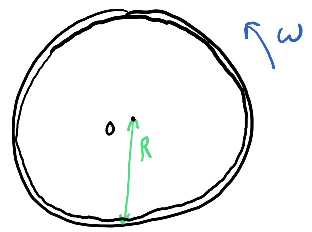
$$\vec{L}_o = I_o\, \vec{\omega}$$

Notice that this new form of angular momentum as the same functional form as translational momentum. Translational momentum was defined as velocity scaled by mass, where angular momentum in this form is defined as angular velocity scaled by moment of inertia. We also now know that moment of inertia plays a similar role as mass does in Newton's 2nd law of motion.

"MOMENTUM"		"SCALAR"	"VELOCITY"
$\vec{L}_o$	$=$	I_o	$\vec{\omega}$
$\vec{p}$	$=$	m	$\vec{V}$

PRACTICE: A circular ring of mass m and radius R is rotating counterclockwise at a constant 60 rpm. What is the angular momentum of this ring?

1. 0 m R^2
2. m R^2
3. 3.14 m R^2
4. 6.28 m R^2



$$f = \frac{60\text{ REV}}{\text{MIN}} \times \frac{1\text{ MIN}}{60\text{ SEC}} = \frac{1\text{ REV}}{\text{SEC}}$$

$$L_o = I_o\,\omega \quad I_o \text{ FOR RING} = mR^2$$

$$L_o = mR^2\omega$$

$$L_o = mR^2(2\pi f)$$

$$L_o = 2\pi mR^2 f$$

Conservation of angular momentum

Recall back to our introduction to momentum-impulse theorem. We algebraically manipulated Newton's 2nd law and found an expression that we called the "momentum-impulse" theorem. Below I go through that process again, but next to it I will follow the same procedures but with the rotational version of Newton's 2nd law.

MOMENTUM - IMPULSE

$$\sum \vec{F}_{\substack{EXT\\ \text{ON SYS}}} = M_{SYS}\,\vec{a}_{cm}$$

DEFINITION OF $\vec{a}$

$$\sum \vec{F}_{\substack{EXT\\ \text{ON SYS}}} = M_{SYS}\,\frac{\Delta\vec{v}}{\Delta t}$$

DEFINITION OF $\vec{P}_{SYS}$

$$\sum \vec{F}_{\substack{EXT\\ \text{ON SYS}}} = \frac{\Delta\vec{P}_{SYS}}{\Delta t}$$

$$\underbrace{\sum \vec{F}_{\substack{EXT\\ \text{ON SYS}}}\,\Delta t}_{\text{IMPULSE}} = \Delta\vec{P}_{SYS}$$

ANGULAR MOMENTUM - ANGULAR IMPULSE

$$\sum \vec{\tau}_{\substack{EXT,o\\ \text{ON System}}} = I_{SYS,o}\,\vec{\alpha}$$

DEFINITION OF $\vec{\alpha}$

$$\sum \vec{\tau}_{\substack{EXT,o\\ \text{ON SYS}}} = I_{SYS,o}\,\frac{\Delta\vec{\omega}}{\Delta t}$$

DEFINITION OF $\vec{L}_{SYS,o}$

$$\sum \vec{\tau}_{\substack{EXT,o\\ \text{ON SYS}}} = \frac{\Delta\vec{L}_{SYS,o}}{\Delta t}$$

$$\boxed{\underbrace{\sum \vec{\tau}_{\substack{EXT,o\\ \text{ON SYS}}}\,\Delta t}_{\text{ANGULAR IMPULSE}} = \Delta\vec{L}_{SYS,o}}$$

I've boxed the new rotational version of momentum-impulse since we have seen the translational versions before. The form of the angular momentum angular impulse theorem helps illuminate the idea that a net external torque about some axis "o" acting for some amount of time causes a change in angular momentum.

Study the angular momentum angular impulse theorem carefully. Notice that if the net external torque acting on the system is zero then the change in angular momentum of the system is zero. Thus angular momentum of the system is conserved (initial angular momentum is equal to final angular momentum).

$$\Delta\vec{L}_{SYS,o} = \cancel{\sum \vec{\tau}_{\substack{EXT,o\\ \text{ON SYS}}}}^{\,0}\,\Delta t$$

$$\Delta\vec{L}_{SYS,o} = \vec{0}$$

* CONSERVATION OF

$$\Delta L_{sys,o} = 0$$

$$\boxed{\underset{\text{INITIAL}}{\vec{L}_{sys,o}} = \underset{\text{FINAL}}{\vec{L}_{sys,o}}}$$

* CONSERVATION OF ANGULAR MOMENTUM

In reality, it is very hard to isolate a system such that the net external torque is zero. Thus look for the same approximations as you did with conservation of translational momentum. Namely: if the external torque is very small ; if the time that the net external torque acts on the system is small ; or if both are small, then the change in angular momentum is approximately zero as shown below.

IF $\sum \vec{\tau}_{EXT,o \text{ ON SYS}} \approx \vec{0}$

THEN $\Delta \vec{L}_{sys,o} = \Delta t$

$\Delta \vec{L}_{sys,o} \approx \vec{0}$

OR

IF $\Delta t \approx 0$

THEN $\Delta \vec{L}_{sys,o} = \sum \vec{\tau}_{EXT,o \text{ ON SYS}} \Delta t$

$\Delta \vec{L}_{sys,o} \approx \vec{0}$

OR

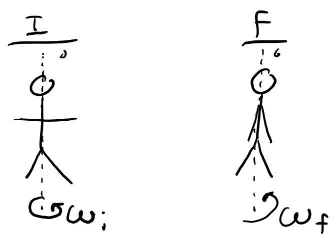
IF $\sum \vec{\tau}_{EXT,o \text{ ON SYS}} \approx \vec{0}$ AND $\Delta t \approx 0$

THEN $\Delta \vec{L}_{sys,o} =$

$\Delta \vec{L}_{sys,o} \approx \vec{0}$

PRACTICE: When an ice skater is spinning and pulls her hands and (free) foot to her body, her angular velocity (blank) because her (blank).

1. increases ; mass decreases
2. decreases ; moment of inertia increases
3. increases ; moment of inertia decreases
4. increases ; moment of inertia increases



$\Delta L_o = \sum \cancel{\tau_o}^{\,0} \Delta t$

$L_{oi} = L_{of}$

$I_i \omega_i = I_f \omega_f$

$\omega_f = \dfrac{I_i}{I_f} \omega_i$ //

MORE MASS CLOSER TO "O" @ FINAL

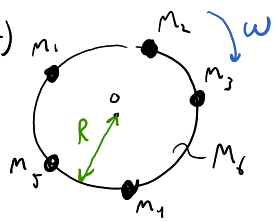
THUS $I \downarrow$

IF $I \downarrow$ THEN $\omega \uparrow$

PRACTICE: A non-powered low friction (100 kg) merry-go-round has five horses each of mass 50 kg and spaced at regular intervals around a circle of radius 3 m. The merry-go-round is rotating clockwise at 1 rpm.

a. What is its angular momentum?
b. If a man of mass 100 kg jumps onto one of the horses, what will be the new angular speed?

a)



$M_1 = M_2 = M_3 = M_4 = M_5$

$$\Sigma L_{o_i} = 5L_{o_{1_i}} + L_{o_{6_i}}$$

$$\Sigma L_{o_i} = 5 I_{o_1}\omega_i + I_{o_6}\omega_i$$

$$\Sigma L_{o_i} = 5mR^2\omega_i + \tfrac{1}{2}M_6\omega_i$$

$$f = \frac{1\ \text{REV}}{\text{MIN}} \times \frac{1\ \text{min}}{60\ \text{SEC}} = \frac{1}{60}\ \frac{\text{REV}}{\text{SEC}}$$

$$\Sigma L_{o_i} = 5mR^2(2\pi f_i) + \tfrac{1}{2}M_6(2\pi f_i) \approx 282.7\ \frac{\text{Nm}}{\text{s}^2}$$

b) $\Delta L_o = \cancel{\Sigma \tau_o}^{\,0}\,\Delta t$ MAN $\rightarrow M_7 = 100\text{ kg}$ @ R From "o"

$$L_{o_i} = L_{of}$$

$$282.7\ \frac{\text{Nm}}{\text{s}^2} = 5mR^2\omega_f + \tfrac{1}{2}M_6R^2\omega_f + m_7R^2\omega_f$$

$$282.7 = 3600\,\omega_f$$

$$\omega_f \approx 0.0785\ \frac{\text{RAD}}{\text{SEC}}$$

or

$$\omega = 2\pi f$$

$$f = 0.0125\ \frac{\text{REV}}{\text{SEC}} \times \frac{60\text{ s}}{1\text{ MIN}}$$

$$f = 0.75\ \frac{\text{REV}}{\text{MIN}}$$

Questions for discussion:

(1) Can an point particle traveling in a straight line have angular momentum?