

## Moment of inertia

### Select LEARNING OBJECTIVES:

- Construct the moment of inertia for rigid objects.
- Estimate the relative moment of inertias for various objects.
- Apply the parallel axis theorem to calculate the moment of inertia of an object about a new location.

### TEXTBOOK CHAPTERS:

- Giancoli (Physics Principles with Applications 7<sup>th</sup>) :: N/A
- Knight :: 12.4
- BoxSand :: Rotational Mechanics ( N/A )

**WARM UP:** Consider the person shown below. In which scenario does it take more effort for the person to begin to rotate?

CASE A

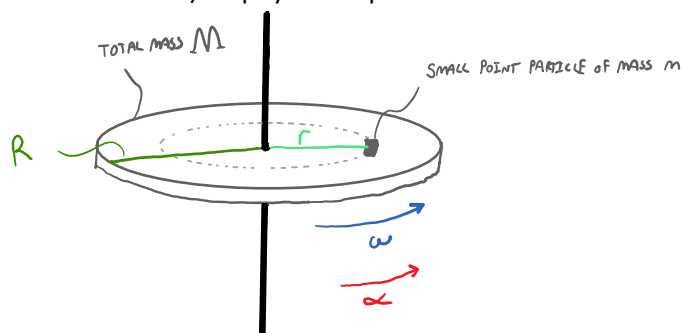


CASE B



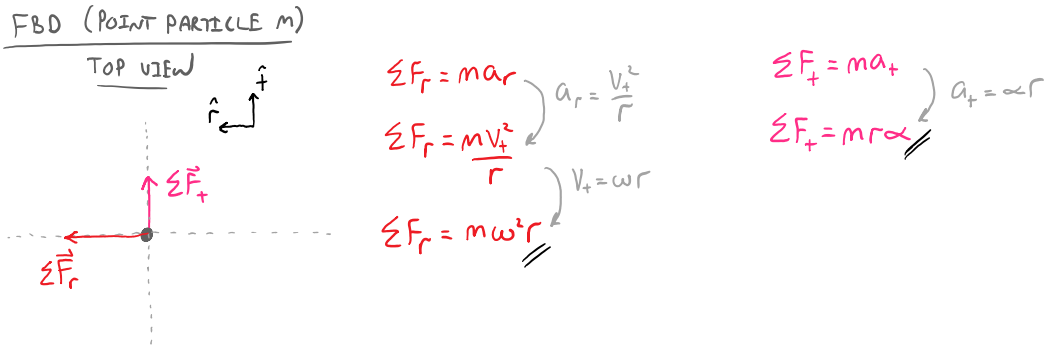
We introduced the concept of moment of inertia in the statics lecture. Recall that statics didn't require knowledge of the moment of inertia to analyze scenarios where objects were not rotating (i.e. rotational static equilibrium). Since we now wish to analyze rotating objects, we will look more closely at moment of inertia before continuing. To do this, we need to start from the beginning (i.e. let's pretend we did not know that  $\Sigma \vec{\tau}_{ext,o} = I_o \vec{\alpha}$ ).

Consider a solid disk of total mass  $M$ , fixed to an axel which it is free to rotate about. Although this example is highly symmetric, it need not be, an analysis of an odd shaped object rotating around a random axis would also work result in the same conclusions. Let's consider the situation where the solid disk is rotating CCW and speeding up (i.e. it has an angular acceleration). A physical representation of this scenario is shown below.

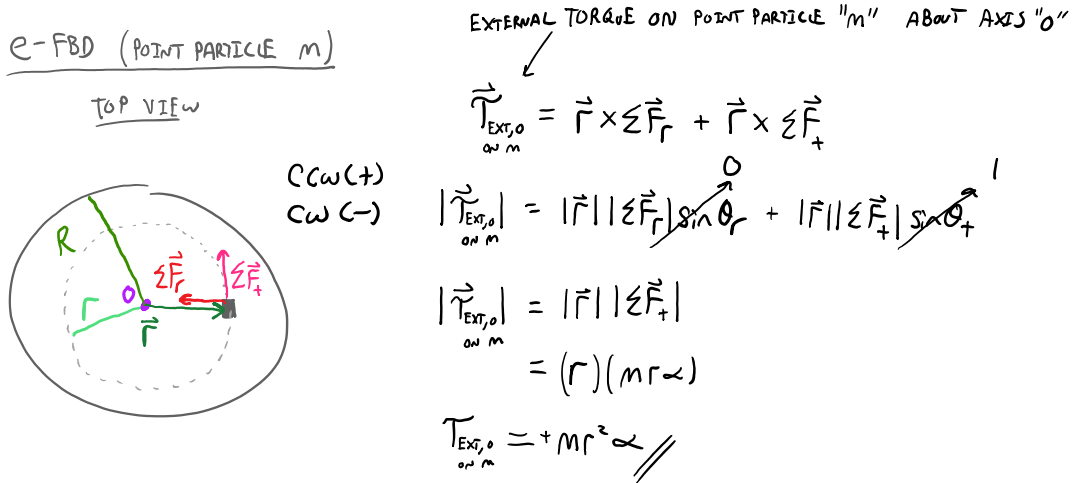




Now let's follow our spidey senses and do what we know, apply Newton's 2nd law to a small segment (i.e. a point particle) of mass "m" of the rotating speeding up disk. We will make use of polar coordinates, thus we have a radial direction and a tangential direction.



As shown above, a free body diagram from a top down perspective indicates two forces acting on the point particle of mass m; one in the radial direction (since the mass is moving in a circle there must be a radial acceleration and thus a net radial force), and one in the tangential direction (since the disk is speeding up there must be a tangential acceleration and thus a net tangential force). Using the free body diagram and Newton's 2nd law, two expressions can be constructed for the net radial force and the net tangential force as shown above. Since these two net forces are acting on the point particle at some distance from a rotational axis, it is natural to look at the torque on the point particle as shown below.



As shown above, extended free body diagrams are useful tools to help get a visual reference to how the forces and moment arms are oriented with respect to one another. Here, there is only one moment arm that points to the location of both net forces acting on the point particle of mass m. The torque on the point particle is then the torque from the net radial force plus the torque from the net tangential force, the result indicates that the torque acting on the point particle of mass m is  $m r^2 \alpha$ .

Note that we did the above analysis for just one point particle of mass m, but our disk is made up of many point particles of mass m, thus the total net torque on the disk (a rigid body), is

$$\sum \vec{\tau}_{\text{Ext},o} = \sum_i^N m_i r_i^2 \alpha$$

NET EXTERNAL TORQUE ON DISK ABOUT AXIS "o"

MOMENT OF INERTIA ABOUT AXIS "o"

$$\sum_i^N m_i r_i^2 \equiv I_o$$

$$\sum \vec{\tau}_{\text{Ext},o} = I_o \alpha$$

And there it is, the expression that I asserted was true in the statics lecture is the end result of our analysis here. The key take away items are the following: a net external torque causes an angular acceleration (this was asserted before, but now you see that it is nothing new; the cause effect relationship literally pops out from a Newtonian analysis, which is why it is often referred to as the rotational version of Newton's 2nd law), and the net external torque is proportional to the resulting angular acceleration by a constant called the moment of inertia which has a mathematical form shown above in orange.

We can make an analogy here; the moment of inertia plays the same role as mass does in the translational version of Newton's 2nd law (shown below).

TRANSLATIONAL 2<sup>nd</sup> LAW

$$\sum \vec{F}_{\text{EXT}} = M_{\text{sys}} \vec{a}_{\text{cm}}$$

↑ CAUSE

↑ PROPORTIONALITY CONSTANT

↑ EFFECT

ROTATIONAL 2<sup>nd</sup> LAW

$$\sum \vec{\tau}_{\text{EXT},o} = I_{o,\text{sys}} \alpha$$

↑ CAUSE

↑ PROPORTIONALITY CONSTANT

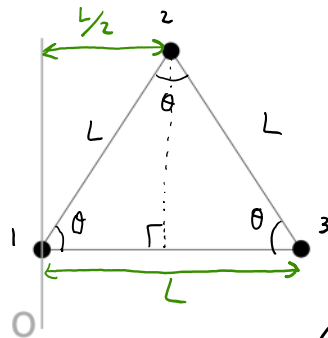
↑ EFFECT

Unfortunately determining the moment of inertia is not as easy as determining the mass (to find the mass grab the nearest scale and directly measure it). To determine the moment of inertia we must do calculations. If you have a finite number of point particle objects, follow the recipe that fell out of the analysis done in the beginning

$$I_o = \sum_i^N m_i r_i^2$$

**PRACTICE:** Consider 3 point particles each of mass  $m$  arranged in the configuration shown below (equilateral triangle of side lengths  $L$ ). Find the moment of inertia about the axis labeled "o".

1.  $m L^2$
2.  $1/2 m L^2$
3.  $3/4 m L^2$
- ④  $5/4 m L^2$



$$I_o = \sum_i^3 m_i r_i^2$$

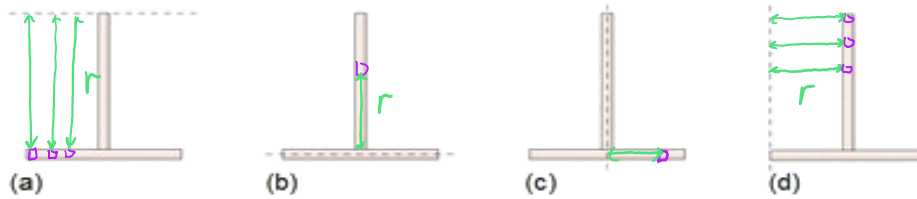
$$= m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2$$

$$= m \left(\frac{L}{2}\right)^2 + m(L)^2$$

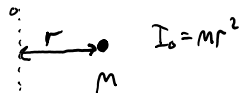
$$= \frac{1}{4} m L^2 + m L^2$$

$$I_o = \frac{5}{4} m L^2$$

**PRACTICE:** Four "Ts" are made from two identical rods of equal mass and length. Rank in order, from largest to smallest, the moment of inertia,  $I_a$  to  $I_d$ , for rotation about the dotted line.



$I_0$  FOR POINT PARTICLE



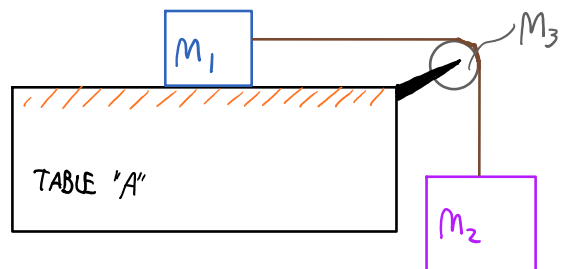
$$I_a > I_d > I_b > I_c$$

**PRACTICE:** Two masses are connected by string across a pulley, only the pulley now has mass, comparably different than when we studied these systems before. Which of the following do you agree with the most?

- F 1. Since the pulley has mass you can just add that extra mass to the hanging mass to account for it.
- F 2. No, since the pulley doesn't move you need to add that mass to the mass on the horizontal surface.
3. I don't know guys, it seems that the pulley must be treated separately and it having mass will change the tension on each side of it. *no*
- F 4. No, the tensions are always the same throughout a rope, what changes is just the need for a torque analysis of the pulley. *yes*

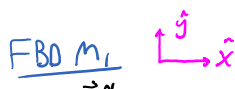
**PRACTICE:** A 4 kg block lies on a horizontal table whose coefficient of friction with the block is 0.30. A horizontal rope is redirected by a uniform solid disk (pulley) of mass 2 kg to a mass of 12 kg hanging off of the table. The radius of the solid disk is 10 cm and the rope doesn't slip on the disk.

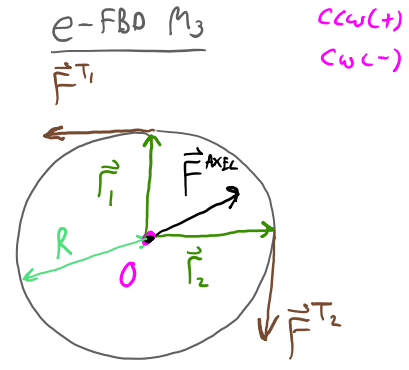
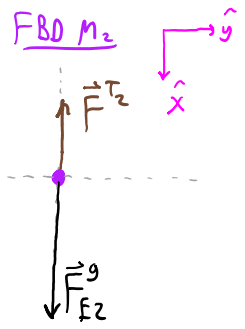
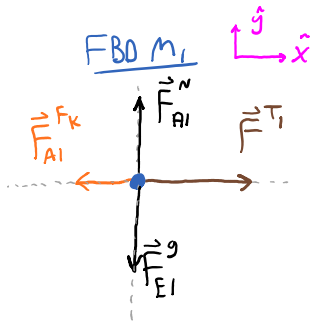
1. What is the translational acceleration of the 4 kg block?
2. What is the angular acceleration of the disk?
3. What is the tension in the rope?
4. How far does the top block travel in 2 seconds?
5. How many revolutions does the disk of the pulley make in 2 seconds?
6. How far does a point on the edge of the disk of the pulley travel in 2 seconds?



CONNECTIONS:  $a_{1x} = a_{2x} \equiv a_x$

$a_x = \alpha R$  (1 Eqn) (5 unknowns)





$$\sum F_x = m_1 a_{1x} \quad \sum F_y = m_1 a_{1y} = 0 \quad \sum F_y = m_2 a_{2x} \quad \sum \tau_o = I_o \alpha$$

$$|\vec{F}^{T1}| - |\vec{F}_{AI}^{Fk}| = m_1 a_x \quad |\vec{F}_{AI}^N| - |\vec{F}_{EI}^g| = 0 \quad |\vec{F}_{E2}^g| - |\vec{F}^{T2}| = m_2 a_x \quad |\vec{r}_1| |\vec{F}^{T1}| \sin \theta_1 - |\vec{r}_2| |\vec{F}^{T2}| \sin \theta_2 = -\frac{1}{2} m_3 R^2 \alpha$$

$$|\vec{F}^{T1}| - \mu_k |\vec{F}_{AI}^N| = m_1 a_x \quad |\vec{F}_{AI}^N| - m_1 g = 0 \quad m_2 g - |\vec{F}^{T2}| = m_2 a_x \quad R |\vec{F}^{T1}| - R |\vec{F}^{T2}| = -\frac{1}{2} m_3 R^2 \alpha$$

$$|\vec{F}^{T1}| - |\vec{F}^{T2}| = -\frac{1}{2} m_3 \alpha$$

5 Eqs  
5 unknowns

SUBTRACT

$$|\vec{F}_{AI}^N| = m_1 g$$

$$|\vec{F}^{T2}| = m_2 g - m_2 a_x$$

$$|\vec{F}^{T1}| - \mu_k m_1 g = m_1 a_x$$

$$|\vec{F}^{T1}| = m_1 a_x + \mu_k m_1 g$$

$$|\vec{F}^{T1}| - |\vec{F}^{T2}| = (m_1 a_x + \mu_k m_1 g) - (m_2 g - m_2 a_x)$$

$$|\vec{F}^{T1}| - |\vec{F}^{T2}| = m_1 a_x + \mu_k m_1 g - m_2 g + m_2 a_x$$

$$(m_1 + m_2) a_x + (\mu_k m_1 - m_2) g = -\frac{1}{2} m_3 R \alpha$$

$$(m_1 + m_2) a_x + (\mu_k m_1 - m_2) g = -\frac{1}{2} m_3 R \left( \frac{a_x}{R} \right)$$

$$(m_1 + m_2) a_x + \frac{1}{2} m_3 a_x = -(\mu_k m_1 - m_2) g$$

$$a_x \left( m_1 + m_2 + \frac{1}{2} m_3 \right) = (m_2 - \mu_k m_1) g$$

1)  $a_x = \frac{(m_2 - \mu_k m_1) g}{(m_1 + m_2 + \frac{1}{2} m_3)} \approx 6.23 \text{ m/s}^2$

2)  $\alpha = \frac{a_x}{R} \approx 62.3 \text{ rad/s}^2$

3)  $|\vec{F}^{T1}| = m_1 a_x + \mu_k m_1 g \approx 36.7 \text{ N}$   
 $|\vec{F}^{T2}| = m_2 g - m_2 a_x \approx 42.9 \text{ N}$

4)  $x_f = x_i + v_{ix} \Delta t + \frac{1}{2} a_x \Delta t^2$   
 $\Delta x = \frac{1}{2} a_x \Delta t^2 \approx 12.5 \text{ m}$

5)  $\theta_f = \theta_i + \omega_i \Delta t + \frac{1}{2} \alpha \Delta t^2$

$$\Delta\theta = \frac{1}{2} \omega \Delta t^2$$

$$1 \text{ REV} = 2\pi \text{ RAD}$$

$$\Delta\theta \times \frac{1 \text{ REV}}{2\pi \text{ RAD}} = \frac{124.6447054}{2\pi} \approx \boxed{19.8 \text{ REvolutions}}$$

6) ARC LENGTH  $\equiv S$

$$S = R\Delta\theta \approx \boxed{12.5 \text{ m}}$$

✓ consistent w/ 4 i