Rotational kinematics

Select LEARNING OBJECTIVES:

i. Characterize motion of an object traveling around a circular path in terms of the natural variables: angular position (radians), angular velocity (radians/time), and angular acceleration (radians/time^2)
ii. Draw physical representations of the objects motion that include rotational kinematics quantities such as angular position, velocity, and acceleration
iii. Organize rotational kinematics variables in terms of known and unknown quantities for a particular problem
iv. Use the appropriate rotational coordinate system and apply the convention of counter-clockwise being the positive direction
v. Apply the rotational kinematics equations for constant acceleration to solve for unknown quantities
vi. Connect rotational quantities to their linear counterpart, e.g. after traveling through a rotational change in position of 1 radian, what is the linear distance traveled
vii. Be able to move between plots of angular position, velocity, and acceleration similarly to the way we do this for linear plots of motion

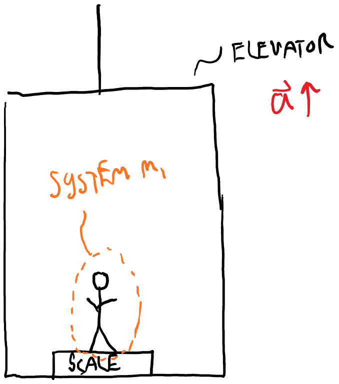
TEXTBOOK CHAPTERS:

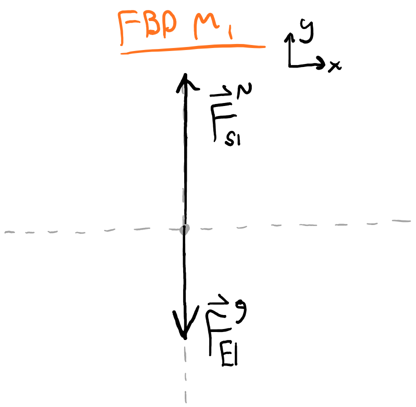
- Giancoli (Physics Principles with Applications 7th) :: 7-1, 7-3
- Knight :: 4.5 - 4.6
- BoxSand :: Rotational mechanics (rotational kinematics)

WARM UP: While in an elevator during the initial speeding upwards phase your apparent weight is 10% larger than your true weight. What is the acceleration of the elevator.

$|\vec{F}^{g}_{E1}|$

WHAT SCALE SHOWS
+ WHAT YOU FEEL
(i.e. THE NORMAL FORCE)





$$\Sigma F_y = m_1 a_y$$

$$|\vec{F}^{N}_{S1}| - |\vec{F}^{g}_{E1}| = m_1 a_y$$

$\uparrow$ SCALE SHOW $|\vec{F}^{N}_{S1}| = (1.1)|\vec{F}^{g}_{E1}|$

$$(1.1)\,|\vec{F}^{g}_{E1}| - |\vec{F}^{g}_{E1}| = m_1 a_y$$

$$1.1\, m_1 g - m_1 g = m_1 a_y$$

$$\boxed{0.1 g = a_y} \approx 1\ \mathrm{m/s^2}$$

Previously we have discussed how to mathematically describe the motion of objects moving in 1-D without concern about what caused the objects to move in the first place; we called this kinematics. We then determined that the horizontal motion of an object does not affect the vertical motion of the object under the constraints we limited ourselves to (e.g. no air resistance). This allowed us to tackle 2-D problems by effectively doing two 1-D problems (one 1-D problem for the horizontal motion, and one 1-D problem for the vertical motion).

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We will now explore a special case of motion where an object moves around in a circle. This circular motion invites us to exploit the symmetries of this system; just like our lecture on uniform circular motion (UCM) we will use polar coordinates. The extra complexity we add here is that we will now allow for a tangential acceleration, thus rotational kinematics is <u>not</u> uniform circular motion.
Recall our discussion about a choice of coordinate system for UCM; Cartesian coordinates involved time dependent x and y coordinates, while polar coordinates allowed us to simplify the problem by reducing the time dependence to only one variable, the angular position (θ). Below is a review of polar coordinates.

Remember the definition of average velocity? It is the change in position divided by the change in time. For rotational kinematics, the change in angular position divided by the change in time is known as the "average angular velocity" which we denote by using the variable, $\overline{\omega}$. Continuing forward with our linear counterparts, the change in velocity divied by the change in time is the average acceleartion. Thus the change in angular velocity divided by the the change in time is the average angular acceleration which is written as $\overline{\alpha}$. To summarize:

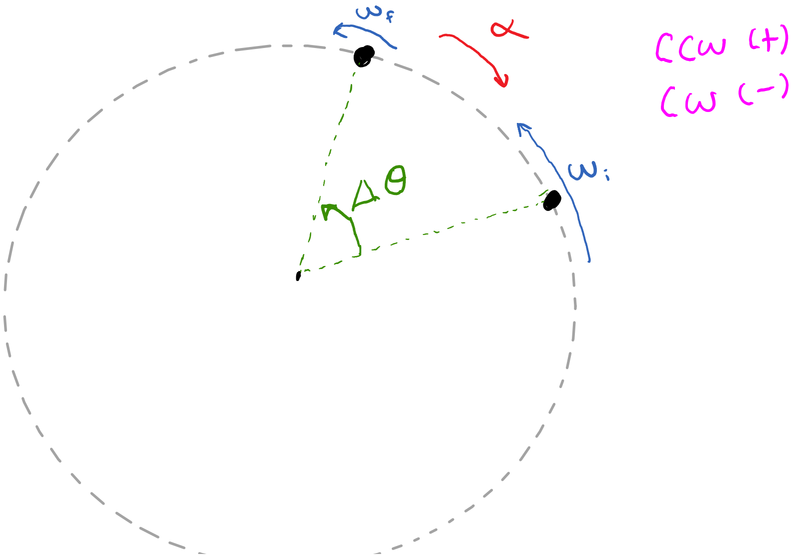
ANGULAR POSITION	AVERAGE ANGULAR VELOCITY	AVERAGE ANGULAR ACCELERATION
θ (RADIANS)	$\overline{\omega} = \frac{\Delta\theta}{\Delta t}$ $\left(\frac{\text{RADIANS}}{\text{SECOND}}\right)$	$\overline{\alpha} = \frac{\Delta\omega}{\Delta t}$ $\left(\frac{\text{RADIANS}}{\text{SECOND}^2}\right)$

Note that the units are in terms of radians, not degrees. Always make sure to use radians for the angular position, so convert any information given in degrees to radians first before applying any of the above definitions.

$$\boxed{2\pi \text{ (RAD)} = 360^\circ}$$

<u>Physical representation</u>

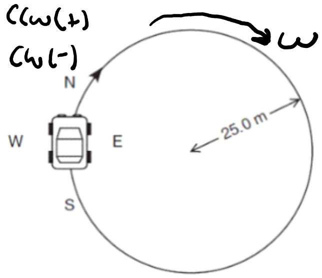
Just like linear kinematics, a physical representation for rotation kinematics is extremely important, it connects the generic variables in a mathematical equation to the physical situation that we are attempting to describe. Our physical representation for rotational kinematics is similar to linear kinematics: we need to identify at least 2 locations (angular positions), the angular velocities at those locations, the angular acceleration between those locations (angular acceleration must be constant between the two locations), a coordinate system, and the change in angular position. An example is shown below.



Notice how the angular velocity and angular acceleration information is given by "curved" arrows. This is sort of misleading, angular velocity and angular acceleration are truly vectors and vectors are represented by straight arrows. So what is going on here? Well, the vector nature of these quantities are beyond the scope of this class at this point. Thus we brush it under the rug and deal with the vector nature of these quantities by assigning the convention of a positive value if it is rotating in the Counter Clock Wise direction, and negative if it is pointing in the Clock Wise direction. Therefore our rotational kinematics coordinate system is this CCW(+) and CW(-) convention. (If you are really interested in which direction these vectors point, the angular velocity vectors for the above physical representation are pointing out of the page, and the angular acceleration vector is pointing into the page).

PRACTICE: A car is traveling clockwise around a circular track, as shown in the figure. If the car is traveling at a constant rate, what is the sign of the car's following angular quantities?

CCW(+)
CW(-)

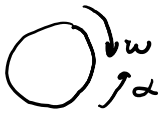


$\Delta\theta$	ω	α
−	−	0

SAME SIGN

$$\omega = \frac{\Delta\theta}{\Delta t}$$

If the car is slowing down, what is the sign of the following angular quantities?

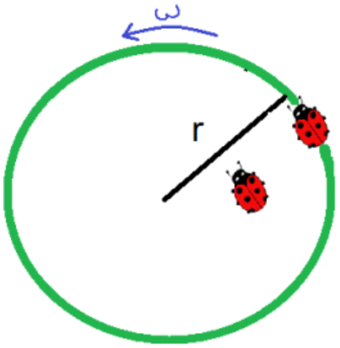


$\Delta\theta$	ω	α
−	−	+

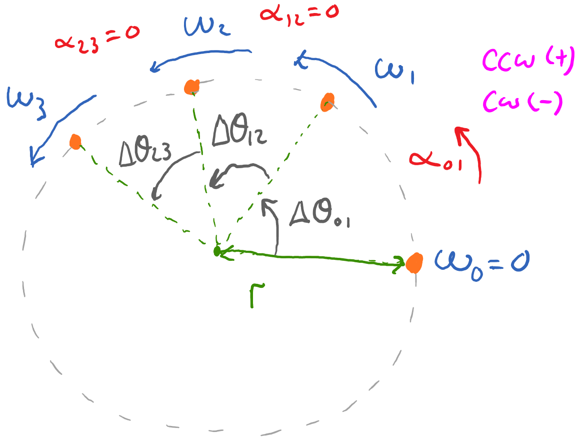
PRACTICE: A ladybug sits on the outer edge of a merry-go-round, and a gentleman bug sits halfway between her and the axis of rotation. The merry-go-round makes a complete revolution once each second. The gentleman bug's angular speed is

(1) greater than the ladybug's.
(2) less than the ladybug's.
[✓ (3)] equal to the ladybug's.
(4) impossible to determine without knowing the radii.

$$\frac{\Delta\theta}{\Delta t} = \omega \leftarrow$$ SAME FOR BOTH

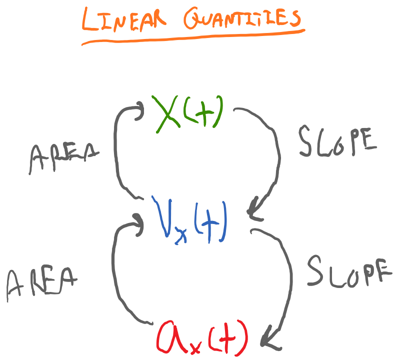


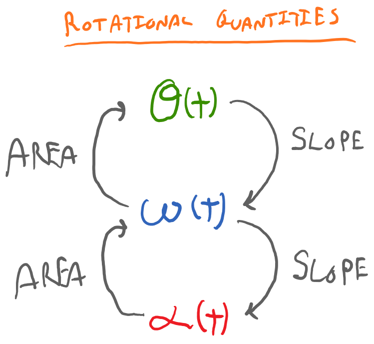
PRACTICE: Draw a physical representation of a CD that starts from rest and speeds up to a constant frequency of 500 RPM.



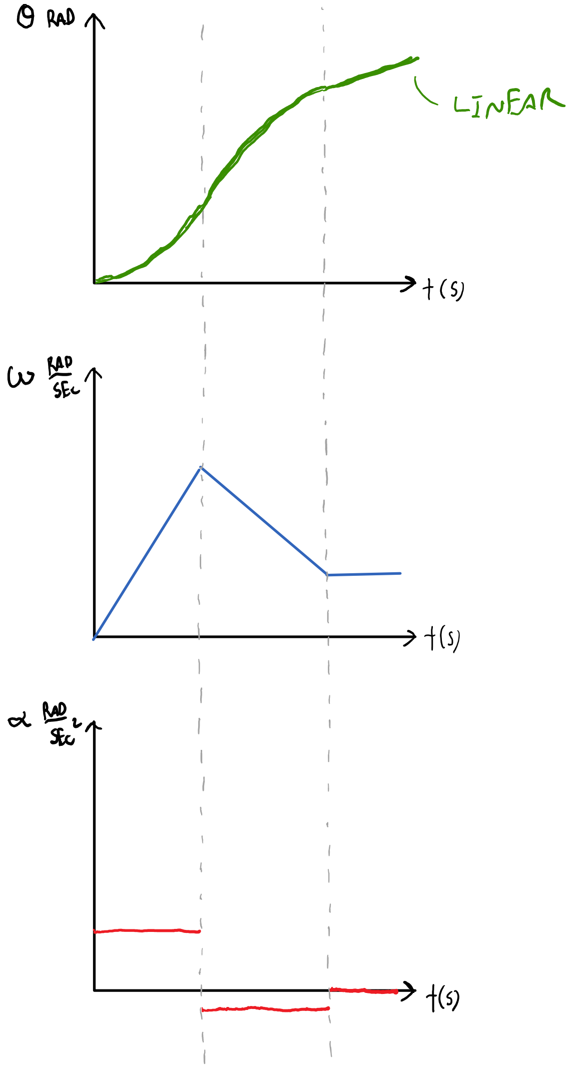
Graphical analysis

The definitions of angular position, average angular velocity, and average angular acceleration should suggest that we can apply the same linear graphical analysis tools; in fact that is correct, the way we move between these angular quantities is directly analogous to their linear counterparts as shown below.



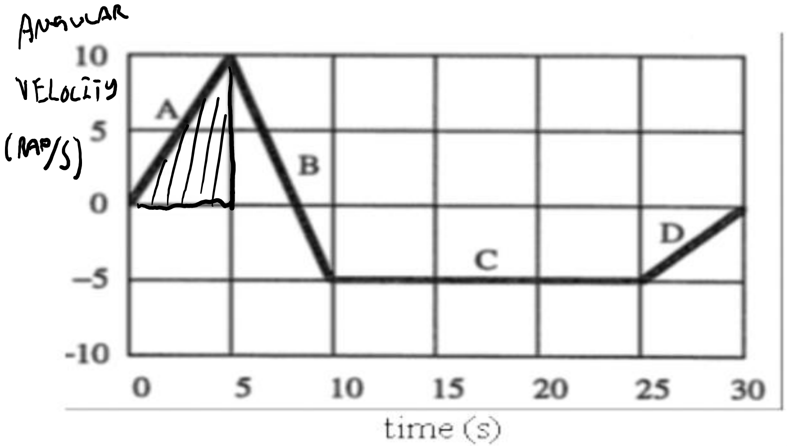


PRACTICE: The graphs below represent the motion of a crankshaft rotating in a car. Sketch the corresponding angular position vs time and angular acceleration vs time graphs. Assume the initial angular position is 0 rad.



PRACTICE: The plot describes the angular velocity versus time for an object moving around a 2-m-radius circle. Four time segments: A, B, C, and D have been labeled. Assume the initial angle is zero.

(a) What is the object's angular acceleration during time segment B (5s < t < 10 s) ?
(b) How many radians does the object subtend (go through) during time segment A (0 s < t < 5 s)?
(c) What is the linear speed of the object during the time segment in which the object is undergoing uniform circular motion?
(d) Extra credit: At what time, other than t = 0 s, is the object's net displacement equal to zero?

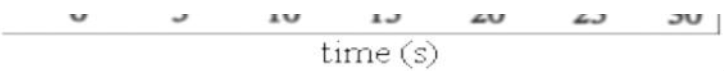


$$\begin{tikzcd} \theta(t) \arrow[d, bend left, "S."] \\ \omega(t) \arrow[u, bend left, "A."] \arrow[d, bend left, "S."] \\ \alpha(t) \arrow[u, bend left, "A."] \end{tikzcd}$$

SLOPE

b) $\omega(t) \xrightarrow{\text{AREA}} \theta(t)$

c) UCM → ω = CONST



a) $\omega(t) \xrightarrow{\text{SLOPE}} \alpha(t)$

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_f - \omega_i}{t_f - t_i}$$

$$= \frac{-5-10}{10-5}\ \frac{\text{RAD}}{s^2}$$

$$\boxed{\alpha = -3\ \text{RAD}/s^2}$$

b) $\omega(t) \xrightarrow{\text{AREA}} \theta(t)$

$$\Delta\theta_A = \text{AREA}$$

$$= \tfrac{1}{2}(5)(10)\ \text{RAD}$$

$$\boxed{\Delta\theta_A = 25\ \text{RAD}}$$

c) $\text{UCM} \rightarrow \omega = \text{CONST}$

$$\omega = -5\ \text{RAD/s}$$

$$V_t = \omega r$$

$$= -5(2)\ \text{m/s}$$

$$\boxed{V_t = -10\ \text{m/s}}$$

Rotational kinematic equations

Recall back to our linear kinematic lectures where we first introduced the definitions of position, average velocity, and average acceleration. Since each quantity is a function of time, we were able to make position vs time, velocity vs time, and acceleration vs time graphs. Finally from the graphs, we derived 3 general equations under the assumption that the acceleration is constant between the two time intervals we chose. We called these 3 equations the kinematic equations; they are the mathematical representation of motion of an object without regard to the cause of the motion. The exact same procedure can be followed with angular position, average angular velocity, and angular acceleration. Since this procedure is exactly the same, I will leave this as an exercise for you to do at home. The end result is 3 rotational kinematic equations as shown below.

$$\theta_f = \theta_i + \omega_i \Delta t + \tfrac{1}{2}\alpha \Delta t^2$$

$$\omega_f = \omega_i + \alpha \Delta t$$

$$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta$$

$\xrightarrow{\text{COMPARE}}$

$\theta \longrightarrow x$

$\omega \longrightarrow V_x$

$\alpha \longrightarrow a_x$

$$x_f = x_i + V_{ix}\Delta t + \tfrac{1}{2} a_x \Delta t^2$$

$$V_{fx} = V_{ix} + a_x \Delta t$$

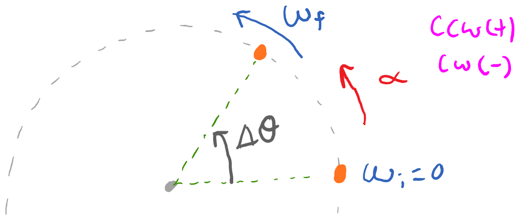
$$V_{fx}^2 = V_{ix}^2 + 2a_x\Delta x$$

Remember that we constrained ourselves to constant angular acceleration between the two time intervals of interest. Also keep in mind that the angular position is to be measured in radians (rad).

PRACTICE: A 40 cm-radius disk begins rotating from rest and spins up uniformly to an angular speed of 4500 RPM in 5 seconds.

$\rightarrow f = 75\ \text{Hz}$

a) What is the angular acceleration (in rad/s^2) of the disk?



$\theta_i = 0$

$\theta_f = ?$

$\omega_i = 0$

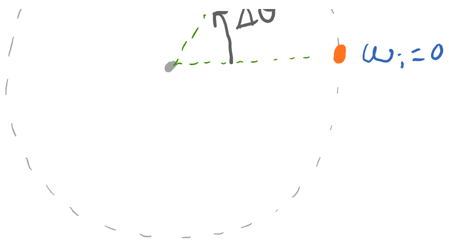
$\omega_f = 2\pi f$

$$\overset{?}{\theta_f} = \overset{\checkmark}{\theta_i} + \overset{\checkmark}{\omega_i}\overset{\checkmark}{\Delta t} + \tfrac{1}{2}\overset{?}{\alpha}\Delta t^2$$

1 EQN
2 UNKNOWNS

$$\overset{\checkmark}{\omega_f} = \overset{\checkmark}{\omega_i} + \overset{?}{\alpha}\overset{\checkmark}{\Delta t}$$

$$2\pi f = \alpha\Delta t$$



$\omega_i =$

$\omega_f = 2\pi f$

$\alpha = ?$

$\Delta t = 5$

$$2\pi f = \alpha \Delta t$$

$$\boxed{\alpha = \frac{2\pi f}{\Delta t}} \approx 94.2 \ \frac{\text{RAD}}{s^2}$$

1. What is the change in angular position (in rad) during the first 3 seconds?

$\theta_i = 0$

$\theta_f = ?$

$\omega_i = 0$

$\omega_f = ?$

$\alpha = 94.2 \ \frac{\text{RAD}}{s^2}$

$\Delta t = 3$

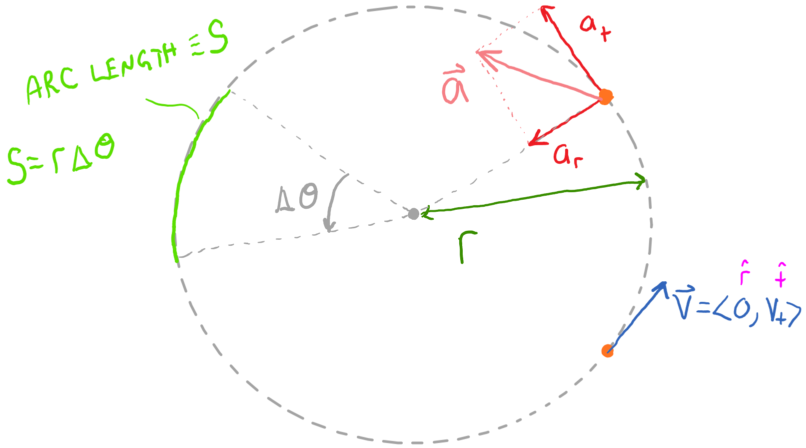
$$\overset{?}{\theta_f} = \overset{\checkmark}{\theta_i} + \overset{\checkmark}{\omega_i} \overset{\checkmark}{\Delta t} + \tfrac{1}{2} \overset{\checkmark}{\alpha} \Delta t^2$$

FROM PART a

$$\boxed{\theta_f = \tfrac{1}{2}\alpha \Delta t^2} \approx 424 \text{ RAD}$$

Connecting rotational quantities to their polar coordinates

The rotational quantities we have discussed so far in this lecture can be related to polar coordinates. While studying UCM in previous lectures, we introduced polar coordinates which had a radial ($\hat{r}$) component and a tangential ($\hat{t}$)component. We also ran into some complications when trying to draw UCM FBDs when viewing the scenario from a top down view or side view. To help clarify this confusion we needed to introduce a 3rd component that was perpendicuarl to both the raidal and tangental components; we called this the z-component ($\hat{z}$). (If we include the z-component we technically don't call it polar coordinates anymore, it is cylindrical coordinates.) Our goal now is to relate angular position, angular velocity, and angular acceleration to polar coordinates. We have done this for UCM but with the addition of a tangential acceleration, it is no longer UCM, it is called non-uniform circular motion.



From the above picture we can state the following definitions.

DISTANCE TRAVELED (ARC LENGTH)

$$S = r\Delta\theta \qquad (\text{METERS})$$

SPEED

$$\frac{\text{DISTANCE}}{\text{TIME}} = \frac{S}{\Delta t} = r\frac{\Delta\theta}{\Delta t} \;\dots\; \lim_{\Delta t\to 0} r\frac{\Delta\theta}{\Delta t} = r\frac{d\theta}{dt} = r\omega$$

$$|\vec{v}| = V_t = \omega r \qquad \left(\frac{\text{METERS}}{\text{SECOND}}\right)$$

RADIAL ACCELERATION

$$\boxed{a_r = \frac{V_t^2}{r} = \omega^2 r} \qquad \left(\frac{\text{METERS}}{\text{SECOND}^2}\right)$$

TANGENTIAL ACCELERATION

$$\boxed{a_t = \alpha r} \qquad \left(\frac{\text{METERS}}{\text{SECOND}^2}\right)$$

Since this is rotational motion, we also still have our definitions of period (T) and frequency (f). We can relate angular frequency (ω) to these two quantites as follows.

$$\boxed{\omega = 2\pi f} \qquad \boxed{\omega = \frac{2\pi}{T}}$$

It is nice to compactly write out how we go from angular quantities to polar coordinates in vector form which is shown below.

$$\vec{r} = \langle \overset{\hat{r}}{r}, \theta(t) \rangle$$

CONSTANT

$$|\vec{r}| = r$$

$$\vec{V} = \langle \overset{\hat{r}}{\cancel{V_r}^{\,0}}, \overset{\hat{t}}{V_t} \rangle$$

$$\vec{V} = \langle 0, V_t \rangle$$

$$\vec{V} = \langle 0, \omega r \rangle$$

$$|\vec{V}| = V_t$$

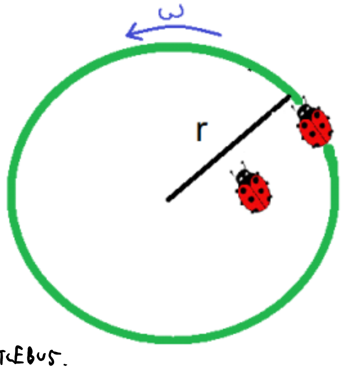
$$\vec{a} = \langle \overset{\hat{r}}{a_r}, \overset{\hat{t}}{a_t} \rangle$$

$$\vec{a} = \left\langle \frac{V_t^2}{r}, \alpha r \right\rangle = \langle \omega^2 r, \alpha r \rangle$$

$$|\vec{a}| = \sqrt{a_r^2 + a_t^2}$$

PRACTICE: A ladybug sits on the outer edge of a merry-go-round, and a gentleman bug sits halfway between her and the axis of rotation. The merry-go-round makes a complete revolution once each second. The gentleman bug's speed is

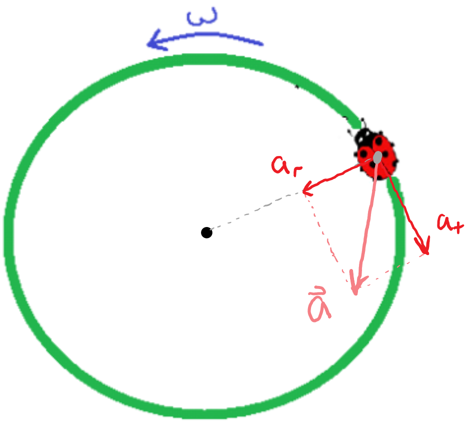
(1) greater than the ladybug's.
[✓ (2)] less than the ladybug's.
(3) equal to the ladybug's.
(4) impossible to determine without knowing the radii.



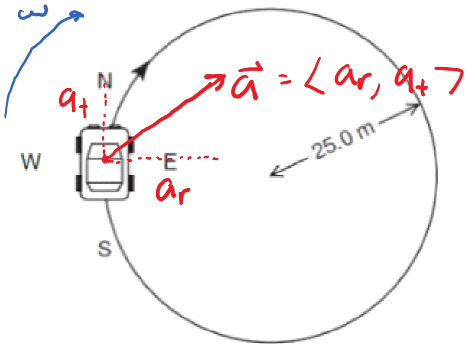
$$\frac{\Delta\theta}{\Delta t} = \omega \leftarrow \text{SAME FOR BOTH}$$

WITH $V_t = \omega r$ AND r IS LESS FOR GENTLEBUG.

PRACTICE: A ladybug sits on the outer edge of a merry-go-round that is turning in the CCW direction and slowing down. At the instant shown in the figure below, draw the radial acceleration, tangential acceleration, and the acceleration vector at the location of the ladybug.



PRACTICE: A car is traveling clockwise around a circular track as shown in the figure. If the car is speeding up, sketch a vector coming off the car in the direction of the car's linear acceleration.



PRACTICE: The acceleration of an object traveling in a circle is comprised of a radial component of acceleration and a tangential component of acceleration. Which of the following pairs of acceleration components represents uniform circular motion?

a. $a_r = 0$, $a_t = 0$ AT REST .... $V_t = 0$ + $\Delta\omega = 0$

[✓ b.] $a_r \neq 0$, $a_t = 0$ UCM ∴ $V_t \neq 0$ BUT $\Delta\omega = 0$

c. $a_r \neq 0$, $a_t \neq 0$ SPEEDING UP OR SLOWING DOWN ∴ $V_t \neq 0$ + $\Delta\omega \neq 0$

$$a_r = \frac{V_t^2}{r} \qquad a_t = \alpha r$$

Questions for discussion

1. Do you agree with the statement: the frequency of a cd disk at the outer edge is larger than at the inner edge?
2. You are driving along a straight horizontal road with the cruise control on (e.g. constant speed). Is it possible to accelerate without stepping on the gas or brake?