Rotational kinetic energy

Select LEARNING OBJECTIVES:

i. Be able to identify whether a system has rotational kinetic energy or not.
ii. Be able to apply conservation of energy to a system with a changing rotational energy.
iii. Demonstrate the ability to apply conservation of energy to a system that has multiple forms of energy including rotational kinetic energy.
iv. Be able to identify how much of the total energy is stored in linear, rotation, and potential energy
    1. Differentiate how much energy goes into rotational kinetic energy based on the moment of inertia of various objects

TEXTBOOK CHAPTERS:

- Giancoli (Physics Principles with Applications 7th) ::
- Knight ::
- BoxSand :: Rotational Mechanics (Rotational energy)

WARM UP: What do you remember about energy?

When we were working under the point particle model (PH201), our progression when something like this:

1. Kinematics (motion without cause)
2. Forces and Newton's laws of motion (motion with cause)
3. Momentum (conservation law, helps simplify studying the motion of objects)
4. Work and energy (conservation law, greatly simplifies analysis of some problems)

We have so far followed this recipe with rotational motion.

1. Rotational kinematics (rotational motion without cause)
2. Torques and the rotational version of Newton's laws (rotational motion with cause)
3. Angular momentum (conservation law)
4. Work and rotational energy

We have covered 1 through 3, which means rotational energy is up next. Recall that energy is a scalar, which is why it makes the math often much simpler than forces or torques which are vectors. In this lecture I hope to do a quick overview of work, introduce rotational kinetic energy, look at objects that are translating and rotating, then finally dive into conservation of energy.

Work

* VALID FOR CONSTANT FORCES AND TORQUES ONLY.

POINT PARTICLE MODEL "TRANSLATIONAL"

$$W = \vec{F} \cdot \Delta\vec{r}$$

RIGID BODY MODEL "ROTATIONAL"

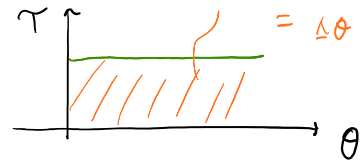
$$W = \tau_o \Delta\theta$$

PRACTICE: The work due to a torque is the area under a

$$W = \tau \Delta\theta$$

1. force vs. position graph.
2. position vs. force graph.
3. torque vs. angular position graph.
4. angular position vs. torque graph.

EXAMPLE



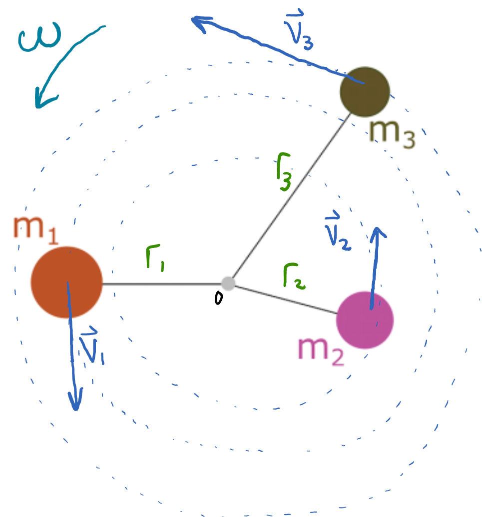
AREA = BASE · HEIGHT

$= \Delta\theta \cdot \tau$

Rotational kinetic energy

Recall the definition of kinetic energy from before (1/2 m v^2), this functional form is the kinetic energy of a point particle as it translates (moves up down left or right). This form served us well when we were ignoring the shape of the objects we were studying, but now we do care about the shape of rigid objects.

Consider a rigid object made from 3 very thin, negligible mass, rods of various lengths rotating ccw about a common axis at a constant angular speed as shown below.



$$KE = \frac{1}{2}m_1|\vec{V}_1|^2 + \frac{1}{2}m_2|\vec{V}_2|^2 + \frac{1}{2}m_3|\vec{V}_3|^2 \qquad |\vec{V}| = \frac{\text{DIST.}}{\text{TIME}} = \frac{2\pi r}{T}$$

$$= \frac{1}{2}m_1\left(\frac{2\pi r_1}{T}\right)^2 + \frac{1}{2}m_2\left(\frac{2\pi r_2}{T}\right)^2 + \frac{1}{2}m_3\left(\frac{2\pi r_3}{T}\right)^2 \qquad T = \frac{1}{f}$$

$$= \frac{1}{2}m_1 r_1^2(2\pi f)^2 + \frac{1}{2}m_2 r_2^2(2\pi f)^2 + \frac{1}{2}m_3 r_3^2(2\pi f)^2$$

$$= \frac{1}{2}m_1 r_1^2\omega^2 + \frac{1}{2}m_2 r_2^2\omega^2 + \frac{1}{2}m_3 r_3^2\omega^2 \qquad \omega = 2\pi f$$

$$= \frac{1}{2}\omega^2\left(m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2\right)$$

$$= \frac{1}{2}\omega^2 \underbrace{\sum_i m_i r_i^2}_{\text{DEFINITION OF MOMENT OF INERTIA}}$$

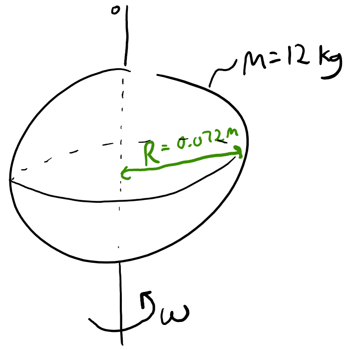
$$\boxed{KE_R = \frac{1}{2} I_{sys,0}\,\omega^2}$$

ANGULAR VELOCITY

ROTATIONAL KINETIC ENERGY

MOMENT OF INERTIA OF OBJECT OR SYSTEM ABOUT AXIS "0"

PRACTICE: A solid 12 kg sphere of radius 7.2 cm is spinning about its center at 100 rpm. What is the rotational kinetic energy of this sphere?



$$KE_R = \frac{1}{2} I_o \omega^2$$

$$= \frac{1}{2}\left(\frac{2}{5}MR^2\right)(2\pi f)^2$$

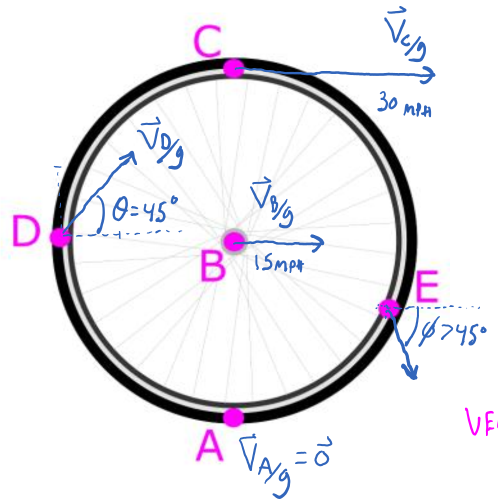
$$KE_R = \frac{4\pi^2 M R^2 f^2}{5} \approx 1.36\ J$$

$$f = 100\ \frac{\text{REV}}{\text{MIN}} \times \frac{1\ \text{MIN}}{60\ \text{SEC}} = \frac{5}{3}\ \text{Hz}$$

Combined rotational and translational motion

Consider a car on a jack stands, we know have the tools necessary to determine the rotational kinetic energy of each wheel as they rotate. Now put the car back onto the ground and drive away at a constant velocity. The sphere is rotating and translating! How do we find the kinetic energy of this complicated system? I'll start by just stating the answer: the total kinetic energy of an object rotating and translating is just the sum of the translational kinetic energy of the center of mass of the object plus the rotational kinetic energy of the object about the axis of rotation. To arrive at this we need to dust off some cobwebs and work with relative velocities.

PRACTICE: A bicycle is traveling at 15 mph along a straight and level road. The tires are rolling without slipping. As you watch the bike go by, sketch the velocity of the tire at the labeled points.



VELOCITY OF i^{th} PARTICLE W.R.T. CENTER OF MASS

$$\vec{V}_{i/g} = \vec{V}_{i/cm} + \vec{V}_{cm/g}$$

$\hat{y}$, $\hat{x}$

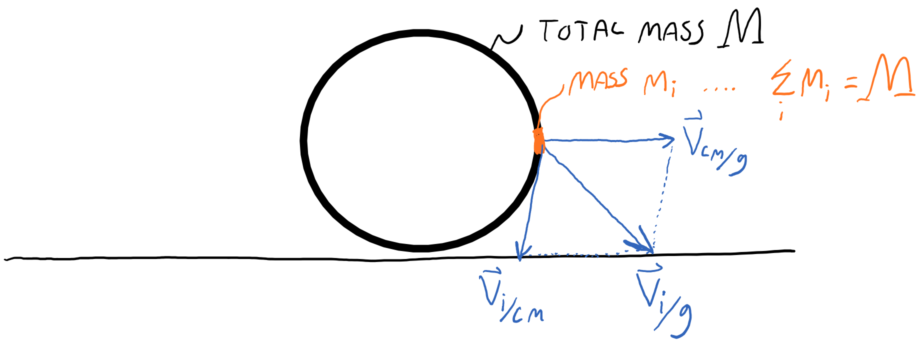
$V_t = \omega r\ \hat{t}$ $15\ \text{MPH}\ \hat{x}$

$15\ \text{MPH}\ \hat{t}$

VELOCITY OF CM W.R.T. GROUND

VELOCITY OF i^{th} LOCATION $\underbrace{\text{WITH RESPECT}}_{\text{W.R.T.}}$ TO THE GROUND

Kinetic energy of wheel rotating and translating



$$KE_{i/g} = \frac{1}{2} m_i \left|\vec{V}_{i/g}\right|^2$$

KINETIC ENERGY OF i^{TH} PARTICLE OF WHEEL W.R.T. GROUND

MASS OF i^{TH} PARTICLE OF WHEEL

SPEED OF i^{TH} PARTICLE OF WHEEL W.R.T THE GROUND

$|\vec{A}| = \vec{A} \cdot \vec{A}$

DOT PRODUCT

$$KE_{i/g} = \frac{1}{2} m_i \; \vec{V}_{i/g} \cdot \vec{V}_{i/g}$$

RE-WRITE VELOCITIES OF EACH i^{TH} PARTICLE W/ RELATIVE VELOCITIES

$$KE_{i/g} = \frac{1}{2} m_i \left(\vec{V}_{i/cm} + \vec{V}_{cm/g}\right) \cdot \left(\vec{V}_{i/cm} + \vec{V}_{cm/g}\right)$$

$$KE_{i/g} = \frac{1}{2} m_i \left(\vec{V}_{i/cm} \cdot \vec{V}_{i/cm} + \vec{V}_{cm/g} \cdot \vec{V}_{cm/g} + 2\,\vec{V}_{i/cm} \cdot \vec{V}_{cm/g}\right)$$

$$KE_{i/g} = \frac{1}{2} m_i \left(\left|\vec{V}_{i/cm}\right|^2 + \left|\vec{V}_{cm/g}\right|^2 + 2\,\vec{V}_{i/cm} \cdot \vec{V}_{cm/g}\right)$$

IN CM FRAME...

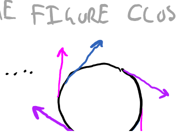
$\left|\vec{V}_{i/cm}\right| = \omega r_{i/cm}$

$$KE_{i/g} = \frac{1}{2} m_i \left((\omega r_{i/cm})^2 + \left|\vec{V}_{cm/g}\right|^2 + 2\,\vec{V}_{i/cm} \cdot \vec{V}_{cm/g}\right)$$

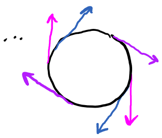
$$KE_{TOTAL/g} = \frac{1}{2} \sum m_i r_{i/cm}^2 \omega^2 + \frac{1}{2} \sum m_i \left|\vec{V}_{cm/g}\right|^2 + \sum m_i \vec{V}_{i/cm} \cdot \vec{V}_{cm/g}$$

$$KE_{TOTAL/g} = \frac{1}{2} \omega^2 \underbrace{\left(\sum m_i r_{i/cm}^2\right)}_{I_{cm}} + \frac{1}{2} \left|\vec{V}_{cm/g}\right| \underbrace{\left(\sum m_i\right)}_{M} + \cancel{\underbrace{\left(\sum m_i \vec{V}_{i/cm}\right)}} \cdot \vec{V}_{cm/g} \rightarrow 0$$

WE ARE SUMMING OVER ALL VELOCITIES OF EACH PARTICLE THAT MAKES UP THE WHEEL W.R.T. THE C.M. LOOK AT THE FIGURE CLOSELY..... THIS SUM IS $\vec{0}$ ....



THIS SUM IS $\vec{0}$ ....



$$\boxed{KE_{TOTAL/g} = \frac{1}{2} I_{cm}\,\omega^2 + \frac{1}{2} M\,|\vec{V}_{cm/g}|^2}$$

IMPLICIT THAT THE FRAME OF REFERENCE IS THE GROUND FRAME FOR KE_{TOTAL} AND $|\vec{V}_{cm}|$

ALSO IMPLICIT THAT KE_R IS ABOUT THE CENTER OF MASS

$$\boxed{KE_{TOTAL} = \underbrace{\frac{1}{2} I_{cm}\,\omega^2}_{KE_R} + \underbrace{\frac{1}{2} M\,|\vec{V}_{cm}|^2}_{KE_{T\,cm/g}}}$$

$$\boxed{KE_{TOT.} = KE_R + KE_T}$$

$$\Delta KE_{sys} = \Delta KE_{T1} + \Delta KE_{T2} + \Delta KE_{R1} + \Delta KE_{R3}$$

$$W^{c}_{INT} \rightarrow -\Delta U^{c}$$

$$-\Delta U^{g} \ldots -\Delta U^{s}$$

$$-\Delta E^{TH}, -\Delta E^{CHEM} \ldots$$

- FRICTION
- HEAT

PEOPLE
ANIMALS
SOME MOTORS...

$$W^{C,NC}_{EXT} = \vec{F}^{\,C,NC}_{EXT} \cdot \Delta\vec{r}$$

$$\boxed{\sum KE_{T_i} + \sum KE_{R_i} + \sum U^g_i + \sum U^s_i + \sum E^{TH} + \sum E^{CHEM}_i + \sum W_{EXT} = \sum KE_{TF} + \sum KE_{RF} + \sum U^g_f + \sum U^s_f + \sum E^{TH}_f + \sum E^{CHEM}_f}$$

ARE ANY OBJECTS IN MY SYSTEM TRANSLATING? MOVING UP DOWN LEFT OR RIGHT?

ARE ANY OBJECTS IN MY SYSTEM ROTATING?

ARE ANY OF THE OBJECTS IN MY SYSTEM CHANGING HEIGHT?

ARE THERE ANY SPRINGS IN MY SYSTEM?

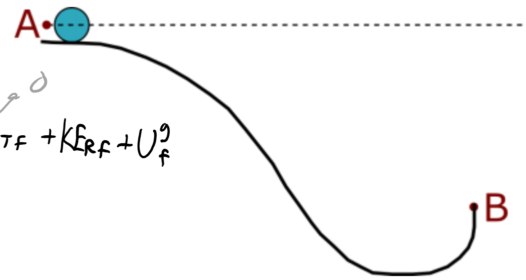
IS FRICTION INTERNAL? ALSO... ANY HEAT TRANSFER MECHANISMS?

ARE THERE ANY LIVING CREATURES OR MOTORS IN MY SYSTEM THAT ARE TRANSFORMING ENERGY WITHIN THE SYSTEM e.g. MOTOR PROTEINS CAR ENGINES, MOTORS

ARE THERE ANY EXTERNAL FORCES DOING WORK ON THE SYSTEM? CAN BE FROM CONSERVATIVE OR NON-CONSERVATIVE FORCES

INCLUDE THESE INTERNAL FORMS OF ENERGY AT THE FINAL STATE

PRACTICE: A solid sphere rolls without slipping along a track shaped as shown below. It starts from rest at point A and is moving vertically when it leaves the track at point B. At its highest point while in the air, will the sphere be __________ point A?



1. above
2. below
3. at the same height as

QUICK

$U^g \longrightarrow KE_T + KE_R$

@ MAX... $KE_T = 0$

$U^g \rightarrow KE_R$

LONG

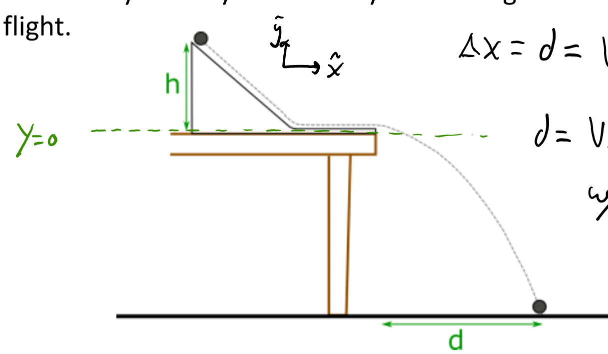
$\cancel{KE_{T_i}}^{0} + \cancel{KE_{R_i}}^{0} + U_i^g + \cancel{W_{EXT}}^{0} = \cancel{KE_{TF}}^{0} + KE_{RF} + U_f^g$

$\underbrace{U_i^g = KE_{RF} + U_f^g}$

$U_i^g > U_f^g$

PRACTICE: A solid sphere, solid disk, and hollow ring of equivalent mass are rolled without slipping down an incline plane on a table. They then fly horizontally off the edge of a table. Rank the horizontal distance each travel during their flight.

- Sphere
- Disk
- Ring



$\Delta x = d = V_{ix}\Delta t + \cancel{\tfrac{1}{2}a_x\Delta t^2}^{0}$

$d = V_{ix}\Delta t$

w/ Δt CONSTANT

$d \propto V_{ix}$

QUICK

$\underbrace{U^g \rightarrow KE_T + KE_R}$ IF $KE_R \uparrow$ THEN $KE_T \downarrow$

$KE_R \propto I_0$

w/ $I_{HOOP} > I_{DISK} > I_{SPHERE}$

$KE_{T_{HOOP}} < KE_{DISK} < KE_{SPHERE}$

w/ m CONST.

$KE_T \propto V^2$

THUS $d_{HOOP} < d_{DISK} < d_{SPHERE}$

LONG

$U_i^g + \cancel{KE_{T_i}}^{0} + \cancel{KE_{R_i}}^{0} + \cancel{W_{EXT}}^{0} = \cancel{U_f^g}^{0} + KE_{T_F} + KE_{RF}$

$U_i^g = KE_{TF} + KE_{TR}$

$KE_{TF} = U_i^g - KE_{TR}$

$KE_{TF} = mgh - \frac{1}{2}I_0\omega$

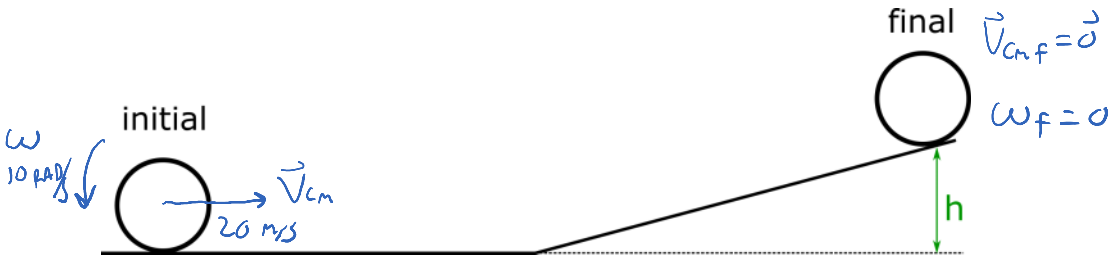
$I_{HOOP} > I_{DISK} > I_{SPHERE}$

w/ m = CONST

$KE_T \propto V^2$

$d_{HOOP} < d_{DISK} < d_{SPHERE}$

PRACTICE: A thin 20 gram hoop or ring with a radius of 2 m is moving so that its center of mass is initially moving at 20 m/s while also rotating without slipping at 10 rad/s along a horizontal surface. It rolls up an incline, coming to rest as shown.



What is the total initial kinetic energy of the hoop?

$$KE = KE_T + KE_R$$

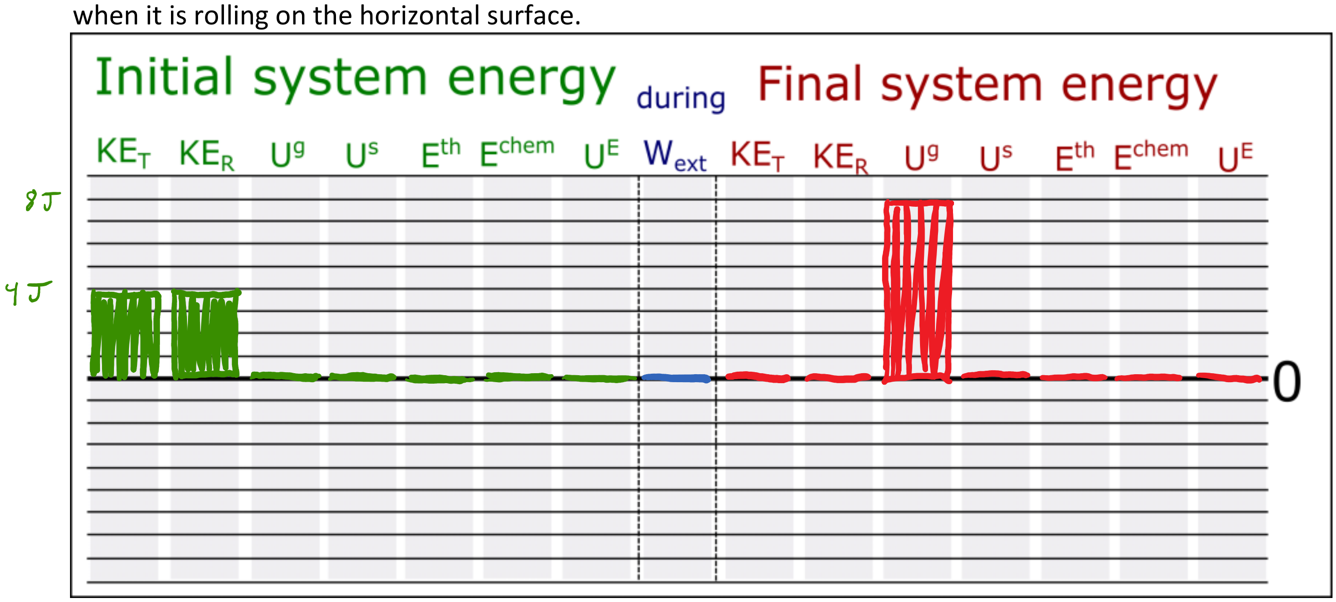
$$KE = \tfrac{1}{2}mV^2 + \tfrac{1}{2}I_{cm}\omega^2$$

$$KE = \tfrac{1}{2}mV^2 + \tfrac{1}{2}mR^2\omega^2$$

$$KE = 4\,J + 4\,J$$

$$KE = 8\,J$$

Complete the qualitative energy bar chart below for the Earth-hoop system for the time between when the hoop is rolling on the horizontal surface and when it has rolled up the ramp and is momentarily at rest. Put the zero point for the gravitational potential energy at the height of the center of the hoop when it is rolling on the horizontal surface.



PRACTICE: A figure skater stands on one spot on the ice (assumed frictionless) and spins around with her arms extended. When she pulls in her arms, she reduces her rotational inertia and her angular speed increases. Compared to her initial rotational kinetic energy, her rotational kinetic energy after she has pulled in her arms must be:

1. the same because no work is done on her.
2. larger because she's rotating faster.
3. smaller because her rotational inertia is smaller.

W/ $\Sigma \tau_{ext,O} = 0$

$\Delta L = 0$

$KE_R = \frac{1}{2} I_o \omega^2$

$L = I_o \omega$

CONSTANT

$KE_R = \frac{1}{2} L \omega$

W/ $\omega \uparrow$

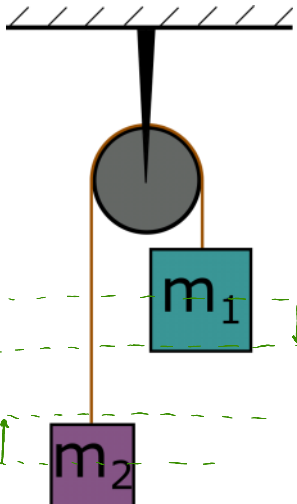
$KE_R \uparrow$

WAIT... WHAT DID WORK ON HER TO INCREASE KE_R?

PERSON... HAS MOTOR PROTEINS WHICH USE PIZZA TO TRANSFORM

$U^{CHEM} \longrightarrow KE_R$

PRACTICE: Two unequal masses are connected across a pulley. A few moments after releasing them from rest, the speed of one of the masses is recorded. If the pulley is replaced with a one of a smaller radius, but equivalent mass, and the experiment is repeated, it is found that the mass is going **faster** after the same amount of time. Why is this true?



Δy_1 Δy_2 $\hat{y}$

SYSTEM (EARTH + $M_1 + M_2 + M_p$)

$U^g_{sys} \longrightarrow KE_{T_1} + KE_{T_2} + KE_{R_p}$

IF $KE_{R_p} \downarrow$

THEN $KE_{T_1} + KE_{T_2} \uparrow$

$KE_{R_p} = \frac{1}{2} I_o \omega^2$

$KE_{R_p} = \frac{1}{2}\left(\frac{1}{2} M_p r^2\right)\omega^2$

SO $KE_{R_p} \propto r^2$

IF $r \downarrow$

THEN $KE_{R_p} \downarrow$

AND $KE_{T_1} + KE_{T_2} \uparrow$

EXAMPLE: $|\Delta y| = 0.5\,m$ $\quad M_p = 100\,g$ $\quad r_p = 0.2\,m$ $\quad |\vec{V}_{f1}| \equiv V_{f1} = ?$

$M_1 = 400\,g$

$M_2 = 200\,g$

$$U^g_{1i} + U^g_{2i} + \cancelto{0}{KE_{T1i}} + \cancelto{0}{KE_{T2i}} + \cancelto{0}{KE_{Rp}} + \cancelto{0}{W_{ext}} = U^g_{1f} + U^g_{2f} + KE_{T1f} + KE_{T2f} + KE_{Rp}$$

$$0 = \Delta U^g_1 + \Delta U^g_2 + KE_{T1f} + KE_{T2f} + KE_{Rp}$$

$$0 = m_1 g \Delta y_1 + m_2 g \Delta y_2 + \frac{1}{2} m_1 V_{1f}^2 + \frac{1}{2} m_2 V_{2f}^2 + \frac{1}{2} I_o \omega_f^2$$

$V_{1f} = V_{2f} \equiv V_f$

$$0 = g\left(m_1 \Delta y_1 + m_2 \Delta y_2\right) + \frac{1}{2} V_f^2 \left(m_1 + m_2\right) + \frac{1}{2}\left(\frac{1}{2} m_p r^2\right)\omega_f^2$$

NO SLIP...

$\omega_f = V_f r$

TANGENTIAL SPEED OF PULLEY

$$0 = g\left(m_1 \Delta y_1 + m_2 \Delta y_2\right) + \frac{1}{2} V_f^2 \left(m_1 + m_2\right) + \frac{1}{4} m_p V_f^2$$

$$0 = g\left(m_1 \underset{(+)}{\Delta y_1} + m_2 \underset{(-)}{\Delta y_2}\right) + \frac{1}{2} V_f^2 \left(m_1 + m_2 + \frac{1}{2} m_p\right)$$

$$\boxed{V_f = 1.74 \text{ m/s}}$$

Questions for discussion:

(1) Discuss the validity of the following statement: if there is no external work done on a system, then the rotational kinetic energy of that system must be constant because of conservation of energy.