Damped oscillations

Select LEARNING OBJECTIVES:

- Understand the definition of a damped oscillation.
- Understand the different types of damped oscillations: underdamped, overdamped, and critically damped.
- Understand the role the damping constant plays in a damped oscillation.
- Be able to use the damping constant to predict features of a damped oscillator

TEXTBOOK CHAPTERS:

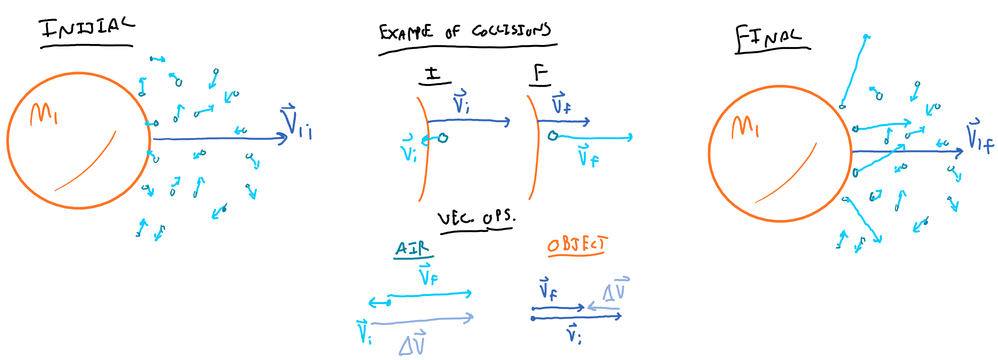
- Boxsand :: [Damping]

WARM UP: A good rule of thumb is that 10° is considered a small angle. Does this mean sin(10°)≈10 ?

Is there something unsettling about our oscillation discussions so far? Set a mass-spring system, pendulum, or really any type of oscillator you can think of into motion and you know that it will eventually come to rest. However, our model for SHO suggested that the motion can be described by sine or cosine functions which continue on forever at time increases. Thus there must be something missing in our previous model. Indeed there is, we assumed no friction or air resistance. In other words, the total energy of oscillators decrease due to the non-conservative work done on the system (i.e. W^{nc} is negative). In this lecture we will introduce how we can include drag (air resistance) within our model of oscillations and also study the consequences of including drag.

Drag

Air is made up of many types of particles, as an object moves through the air the object collides with these particles. Since the air particles have mass the collisions will change the momentum of the particles, and thus also change the momentum of the object. Below is a simplified picture of this collision process.



WITH IMPULSE-MOMENTUM $\Sigma \vec{F} \Delta t = \Delta \vec{P}$

$$\Sigma \vec{F}_{AIR\ on\ OBJECT} = \text{DRAG OR AIR RESISTANCE} \equiv \vec{F}^{D}$$

Note that the collisions between the object and the air particles result in changes of momentum, and that the object's final momentum is lower. The force associated with these collisions is often referred to as drag, or air resistance. From the model above it should be clear that as the objects speed is increased, the change in momentum for the particles is larger, thus a larger change in momentum for the object, resulting in a larger force from the air on the ball (i.e. as speed increases, drag force increases). Thus we can conclude that the drag force is proportional to some function velocity of the object. But what is this function? As it turns out the functional form depends on things like the medium (i.e. the fluid the object is moving through), the object (e.g. object's size, shape, material…). This dependence on so many variables makes the problem very complicated. Luckily, the drag force for many scenarios can be modeled with two common forms as shown below.

LINEAR DRAG FORCE

$$\vec{F}^D = -b\vec{v}$$

CONSTANT
...GIANT RUG WE SWEEP
COMPLICATED PHYSICS UNDER...

QUADRATIC DRAG FORCE

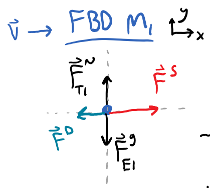
$$\vec{F}^D = -b_2|\vec{v}|^2\,\hat{v}$$

"UNIT VECTOR"
DIRECTION IS THE SAME AS $\vec{v}$
$|\hat{v}| = 1$

For our class, we will only quantify the linear drag force model. Typically, the linear drag force is used for "slow" speeds, but the shape of the object also affects which case should be used (linear or quadratic or both). It is important to note that the drag force always acts in the opposite direction of velocity, which is why there is a negative sign in both models.

Equation of motion

Let's use a mass-spring system to build an equation of motion for damped oscillation. Recall we can build equations of motion by applying Newton's 2nd law. To do this for the spring-mass system with drag, we will consider the mass displaced from its equilibrium location.



$$\Sigma F_x = m_1 a_x$$

$$F^D_x + F^S_x = m_1 a_x$$

$$-b|\vec{v}| - K\Delta X = m_1 a_x \qquad X_{eq}=0$$

$$-b|\vec{v}| - KX = m_1 \frac{\Delta^2 X}{\Delta t^2}$$

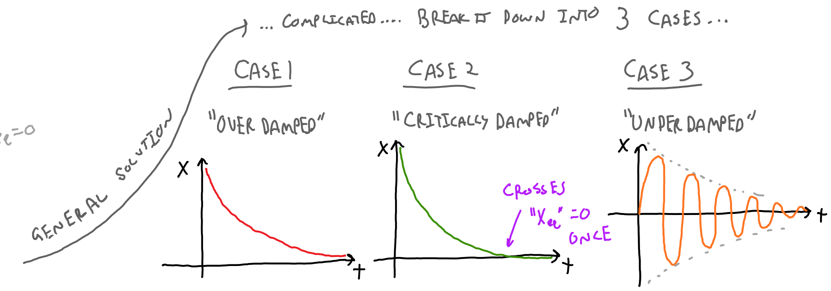
$$\frac{\Delta^2 X}{\Delta t^2} = -\frac{K}{m_1}X - \frac{b}{m_1}|\vec{v}|$$

1-D MOTION $|\vec{v}| = \frac{\Delta X}{\Delta t}$

$$\boxed{\frac{\Delta^2 X}{\Delta t^2} = -\frac{K}{m_1}X - \frac{b}{m_1}\frac{\Delta X}{\Delta t}}$$

CALCULUS

$$\boxed{\frac{d^2x}{dt^2} = -\frac{K}{m_1}X - \frac{b}{m_1}\frac{dx}{dt}}$$



Note that the general solution of motion depends on the relationship between b/m and k/m. For this class we will only look at the underdamped case where the resulting motion is oscillatory but the amplitude decreases as a function of time.

Under damped motion

Note that when a system is underdamped the mass still oscillates back and forth, but as it does the amplitude it reaches after each oscillation gets smaller and smaller. Thus we can guess our general solution for position as a function of time will still have a sine or cosine but will also include another non-oscillating term that decreases in value as a function of time. As it turns out the general solution takes on the form(s) shown below.

$$\boxed{X(t) = \pm \underbrace{X_{max}(t=0)\,e^{-\frac{t}{\tau}}}_{X_{max}(t=0)\,e^{-t/\tau} \,\equiv\, X_{max}(t)}\ \begin{smallmatrix}\text{Sin}\\\text{OR}\\\text{Cos}\end{smallmatrix}(\omega' t)}$$

So...

$$\boxed{X(t) = \pm X_{max}(t)\ \begin{smallmatrix}\text{Sin}\\\text{OR}\\\text{Cos}\end{smallmatrix}(\omega' t)}$$

* WHERE TIME CONSTANT $\equiv \tau = \frac{2m}{b}$

* AND NEW ANGULAR FREQUENCY, $\omega' = \sqrt{\frac{K}{m} - \frac{b}{2m}}$

IF b IS SMALL... $\omega' \approx \omega_0 = \sqrt{\frac{K}{m}}$

NATURAL FREQUENCY

$\omega = 2\pi f = \frac{2\pi}{T}$

$$X(t) = \pm X_{max}(t) \quad (\omega' t)$$

$\omega = 2\pi f = \frac{2\pi}{T}$

$\omega = \sqrt{\frac{k}{m}}$

The important features to observe are that the amplitude $X_{max}(t)$ is now a function of time. The nature of this time dependent amplitude is an exponential decay containing a "time constant" τ which is related to the damping coefficient and mass of the oscillator. Another key feature to make note of is that the angular frequency is slightly different form the natural frequency (the frequency which the mass will oscillate at without damping) but it is still a constant value (i.e. not dependent on time). For our class, we will always assume that the damping constant b is small enough that the resulting oscillatory motion can be described by the natural frequency. Below is a summary of the simplified solutions using this approximation for both a mass-spring and pendulum system.

DAMPED SPRING-MASS SYSTEM

$$X(t) = \pm X_{max}(t=0)\, e^{-t/\tau} \ \begin{matrix}\sin \\ \text{or} \\ \cos\end{matrix} (\omega_0 t)$$

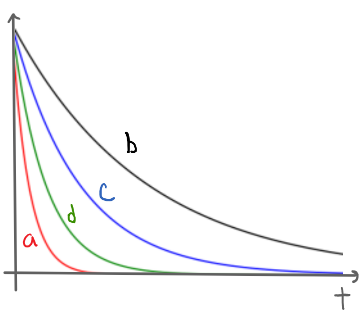
$$\omega_0 = \sqrt{\frac{K}{m}} \qquad \tau = \frac{2m}{b}$$

DAMPED PENDULUM

$$\theta(t) = \pm \theta_{max}(t=0)\, e^{-t/\tau} \ \begin{matrix}\sin \\ \text{or} \\ \cos\end{matrix} (\omega_0 t)$$

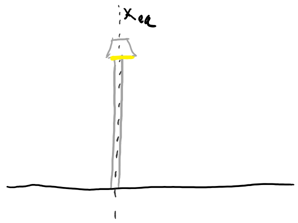
$$\omega_0 = \sqrt{\frac{g}{l}} \qquad \tau = \frac{2m}{b}$$

PRACTICE: Rank in order, from largest to smallest, the time constants τa through τd of the decays below.



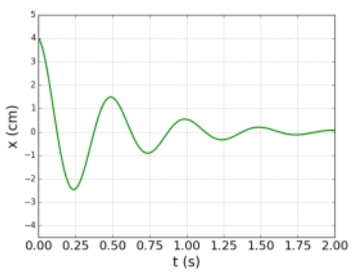
PRACTICE: The amplitude of an oscillator decreases to 10% of its initial value in 10 seconds. What is the value of the time constant.

PRACTICE: A small earthquake starts a lamppost vibrating back and forth. The amplitude of the vibration of the top of the lamppost is 6.5 cm at the moment the quake stops, and 8.0 seconds later it is 1.8 cm. What is the time constant τ fo rthe damping of the oscillation?



What is the amplitude of oscillation at 4.0 seconds after the quake stopped?

PRACTICE: Below is a plot of an underdamped mass-spring system. Estimate the frequency and time constant from this graph.



QUESTIONS FOR DISCUSSION:

(1) The amplitude of an underdamped oscillator is a function of time. Does this function tell you anything about the position of the object at any given time?
(2) The energy of an oscillator decreases with time. Where does this energy go? Describe the energy transformations taking place with a damped system (e.g. damped pendulum or damped mass-spring system).